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Tensor hierarchies of 5- and 6-dimensional field theories

Jelle Hartong[a] **and Tomás Ortín**[b]

[a] *Institute for Theoretical Physics, Sidlerstrasse 5, 3012 Bern, Switzerland*

[b] *Instituto de Física Teórica UAM/CSIC, Facultad de Ciencias C-XVI, C.U. Cantoblanco, E-28049-Madrid, Spain*

E-mail: hartong@itp.unibe.ch, Tomas.Ortin@uam.es

Abstract: We construct the tensor hierarchies of generic, bosonic, 5- and 6-dimensional field theories. The construction of the tensor hierarchy starts with the introduction of two tensors: the embedding tensor ϑ which tells us which vector is used for gauging and another tensor Z which tells us which vector is eaten by a 2-form. In dimensions $d \geq 5$ these two (deformation) tensors are in principle unrelated. Besides ϑ and Z there can be further deformation tensors describing other couplings unrelated to (but compatible with) gauge symmetry. For each deformation tensor there appears a $(d-1)$-form potential and for each constraint satisfied by the deformation tensors there appears a d-form potential in the tensor hierarchy. For each symmetry of the undeformed theory there is an associated $(d-2)$-form appearing in the tensor hierarchy. Our methods easily generalize to arbitrary dimensions and we present a general construction for the d-, $(d-1)$- and $(d-2)$-form potentials for a tensor hierarchy in d dimensions.

Keywords: Field Theories in Higher Dimensions, Gauge Symmetry, Supergravity Models

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Contents

1 Introduction

The structure of the tensor hierarchy[1] of general bosonic 4-dimensional field theories has recently been elucidated in ref. [11] and applied to the search of higher-rank p-form potentials in gauged $N = 1, d = 4$ supergravity in ref. [12].

It is natural to try to extend the recently obtained results on 4-dimensional tensor hierarchies to higher dimensions. The 4-dimensional results suggest the existence of some general features common to all d-dimensional tensor hierarchies:

1. The one-to-one relation between $(d-2)$-form potentials (which always carry an adjoint index) and the symmetries of the theory. We will henceforth refer to them as adjoint-form potentials or simply *ad-form* potentials.

2. The one-to-one relation between the $(d-1)$-form potentials and the components of the embedding tensor (and, possibly, other *deformation tensors*). Following ref. [13], we will call these potentials *de-form* potentials.

3. The one-to-one relation between the top- (d-) form potentials and all the constraints satisfied by the embedding tensor (and, possibly, other deformation tensors).

Some of these relations have been discussed in ref. [14].

In this paper we are going to study in detail 5- and 6-dimensional field theories and we are going to find the general rules that determine the structure of their associated tensor hierarchies. The special case of maximal supergravity in five and six dimensions has been considered in refs. [15, 16].

As we are going to see, there are important differences between the maximal supergravity case and the general case, the principal difference being the existence of more independent deformation tensors in addition to the embedding tensor. These deformation tensors switch on new couplings such as massive deformations, unrelated to (but compatible with) Yang-Mills gauge symmetries, which are determined by the embedding tensor alone. In maximal supergravities, supersymmetry determines these deformation tensors entirely in terms of the gauge group and the embedding tensor. In the general case the deformation tensors are, up to a few constraining relations, independent of the embedding tensor.

Taking into account the existence of several deformation tensors we find that the highest-rank potentials of the tensor hierarchy can be constructed as follows. Let us denote by A^I the 1-forms of the d-dimensional tensor hierarchy, by $\vartheta_I{}^A$ the embedding tensor where A is an adjoint index of some symmetry group and by $c^\sharp$ the deformation tensors (including the embedding tensor). Here $\sharp$ denotes the corresponding indices. The magnetic duals of the 1-forms will be the hierarchy's $(d-3)$-forms $\tilde{A}_I$, with $(d-2)$-form field strengths $\tilde{F}_I$. These will contain a Stückelberg coupling to the ad-form potentials that we are going to denote by C_A, and the coupling tensor will be the embedding tensor $\vartheta_I{}^A$, so

$$\tilde{F}_I \sim \mathfrak{D}\tilde{A}_I + \cdots + \vartheta_I{}^A C_A \,. \tag{1.1}$$

[1]Tensor hierarchies have been introduced in refs. [1–3]. They arise naturally in the embedding tensor formalism [1, 2, 4–6]. For recent reviews see refs. [7–10].

The $(d-1)$-form field strength for C_A, denoted here by G_A, can be obtained by hitting the above expression with a covariant derivative $\mathfrak{D}$. This gives rise to an expression for $\vartheta_I{}^A G_A$ and determines G_A up to terms that vanish upon contraction with $\vartheta_I{}^A$. These extra terms in G_A form Stückelberg couplings to de-form potentials. The coupling tensors will vanish upon contraction (of the adjoint index) with the embedding tensor. They can be constructed in the following way. All the deformation tensors must be gauge-invariant tensors, and, if their gauge transformations are written as

$$\delta_\Lambda c^\sharp = -\Lambda^I Q_I{}^\sharp\,, \tag{1.2}$$

where the $\Lambda^I(x)$ are the 0-form gauge transformation parameters of the 1-forms A^I, then, we find a constraint

$$Q_I{}^\sharp \equiv -\delta_I c^\sharp = 0\,, \tag{1.3}$$

for each of them. All these constraints are, by construction, proportional to the embedding tensor

$$\delta_\Lambda c^\sharp = \Lambda^I \vartheta_I{}^A \delta_A c^\sharp\,, \tag{1.4}$$

and can be written in the form

$$Q_I{}^\sharp = -\vartheta_I{}^A Y_A{}^\sharp\,, \qquad Y_A{}^\sharp \equiv \delta_A c^\sharp\,, \tag{1.5}$$

which provides us with as many tensors $Y_A{}^\sharp$ as we have deformation tensors $c^\sharp$. We will follow the above convention to normalize the constraints Q and associated Y-tensors.

The $(d-1)$-form field strengths will have the form

$$G_A \sim \mathfrak{D}C_A + \cdots + \sum_\sharp Y_A{}^\sharp D_\sharp\,. \tag{1.6}$$

where we have introduced as many de-form potentials $D_\sharp$ as we have deformation tensors $c^\sharp$, transforming in the representation conjugate to the representation in which the $c^\sharp$ transform. This is precisely the number of de-form potentials that we need to introduce in the action as Lagrange multipliers enforcing the constancy of the deformation tensors

$$\int \sum_\sharp dc^\sharp \wedge D_\sharp\,. \tag{1.7}$$

Finally, the d-form field strengths $K_\sharp$ of the de-form potentials $D_\sharp$ will have Stückelberg couplings to top-form potentials. As different from the 4-dimensional case in which there is only one Y-tensor and the Stückelberg coupling tensors (W) are annihilated by the Y-tensor, in the general case the W-tensors are not individually annihilated by the Y-tensors. Instead, there are combinations of Y- and W-tensors that vanish.

These combinations can be found systematically as follows. Let us introduce as many top-form potentials as there are constraints satisfied by the deformation tensors. This is precisely the number of top-forms that we need to introduce in the action as Lagrange multipliers enforcing all the algebraic constraints. We will have top forms $E^I{}_\sharp$ associated to the constraints $Q_I{}^\sharp$ that express the gauge-invariance of the deformation tensors, but we will have more top-forms, associated to other constraints. Let us denote all the constraints satisfied by all the deformation tensors $Q^\flat$ and the top forms by $E_\flat$ and let us construct the formal combination

$$\sum_\flat Q^\flat E_\flat \,, \tag{1.8}$$

which vanishes because it is linear in the constraints. This is the term one needs to add to the action in order to enforce the constraints $Q^\flat = 0$.

The infinitesimal linear transformations of this term generated by the matrices T_A, that we will denote by δ_A, also vanish because these transformations are proportional to the constraints $Q^\flat$. Since the constraints $Q^\flat$ are functions of the deformation tensors, using the chain rule we can write this vanishing infinitesimal transformation as

$$0 = \delta_A \left(\sum_\flat Q^\flat E_\flat \right) = \sum_\flat \left(\sum_\sharp \delta_A c^\sharp \frac{\partial Q^\flat}{\partial c^\sharp} \right) E_\flat = \sum_\flat \left(\sum_\sharp Y_A{}^\sharp \frac{\partial Q^\flat}{\partial c^\sharp} \right) E_\flat \,, \tag{1.9}$$

where we have made use of the general definition of the Y-tensors eq. (1.2). Since, in this expression, the top forms $E_\flat$ have arbitrary values, we get, for each of them, the identity

$$\sum_\sharp Y_A{}^\sharp W_\sharp{}^\flat = 0 \,, \tag{1.10}$$

where we have defined the W-tensors

$$W_\sharp{}^\flat \equiv \frac{\partial Q^\flat}{\partial c^\sharp} \,. \tag{1.11}$$

Then, the d-form field strengths $K_\sharp$ of the de-form potentials $D_\sharp$ will have the general form

$$K_\sharp \sim \mathfrak{D} D_\sharp + \cdots + \sum_\flat W_\sharp{}^\flat E_\flat \,. \tag{1.12}$$

This scheme leads to a number of ad-form potentials C_A equal to the number of (continuous) symmetries and, therefore, to Noether current 1-forms j_A. This is what we expect since, in order not to add further continuous degrees of freedom to the theory the $(d-1)$-form field strengths G_A must be dual to the Noether currents

$$G_A \sim \star j_A \,. \tag{1.13}$$

This scheme also leads to a number of de-form potentials $D_\sharp$ that is equal to the number of deformation tensors $c^\sharp$. As mentioned above, we need this number of deformation tensors to enforce the constraints $dc^\sharp = 0$ in the action. With a Lagrange multiplier term enforcing the constancy of the deformation tensors we can also vary the action with respect to the deformation tensors which have off-shell been promoted to fields. This leads to duality relations for their d-form field strengths $K_\sharp$ of the form

$$K_\sharp \sim \star \frac{\partial V}{\partial c^\sharp} \,. \tag{1.14}$$

Finally, as already said, this scheme leads to one top-form potential for each constraint satisfied by the deformation tensors.

The tensor hierarchy can be considered to be a technique that can be used to predict in which way a given theory can be deformed. To make such a prediction one can construct the de- and top-form field content of a particular theory. The above scheme is only based on necessary conditions and is not guaranteed to be sufficient to construct all possible de- and top-form potentials of a particular (bosonic) field theory.[2] In order to see in which manner the above described construction of the de-forms is not sufficient let us consider possible sources of it failing to be so. For example, it could happen that in order for G_A to transform gauge-covariantly we need to introduce a Stückelberg coupling with a tensor Y_A which is not of the form $\delta_A c$ where c is some deformation tensor but which nonetheless satisfies ${\vartheta_I}^A Y_A = 0$. Even though we have never encountered such a Y-tensor we have not been able to disprove their existence. Similarly, there may be additional top-forms contracted with W-tensors that are not of the form eq. (1.11), but which nonetheless satisfy eq. (1.10). Once again we did not prove that every W-tensor that satisfies eq. (1.10) is of the form eq. (1.11) but we are not aware of any counterexamples. Another source of failure of the above described program to find all the de- and top-form potentials is that there may exist de- and top-form potentials which cannot appear in any Stückelberg couplings. This happens for example in $N = 1$, $d = 4$ supergravity where there exists a 3-form potential that is dual to the superpotential ref. [12]. This 3-form does not show up in any of the Stückelberg couplings of the 4-dimensional tensor hierarchy and there exists no choice of deformations tensors for which it would show up in a Stückelberg coupling.

The construction of any tensor hierarchy starts with writing down the most general form of the 2-form field strength F^I which includes both Yang-Mills pieces as well as Stückelberg couplings to 2-forms. From this field strength, which at this stage should be thought of as an Ansatz, one can construct a Bianchi identity by hitting it with a covariant derivative $\mathfrak{D}$. From $\mathfrak{D}F^I$ we can obtain that part of the field strength of the 2-forms that does not contain the Stückelberg couplings to the 3-forms. By making once again an Ansatz for such a coupling we can proceed to compute the Bianchi identity of the 3-form field strengths and continue in this way until we reach the d-form field strengths of the de-form potentials which contain Stückelberg couplings to the top-form potentials. The Ansätze made throughout this procedure will then lead to a nested set of Bianchi identities provided the various Stückelberg coupling tensors satisfy certain relations. Once these relations have been obtained we have at our disposal the most general set of tensor couplings[3] that a particular bosonic theory can have and we may proceed to construct Lagrangians for these tensors.

This program will be performed in detail in section 2 for the case of 5-dimensional field theory and in the section 3 for the case of 6-dimensional field theory.

[2]When there are also fermions the tensor hierarchy may get extended due to ad-forms that are dual to currents bilinear in fermions that appear in the 1-form equations of motion. These ad-forms may then have Stückelberg couplings with new de-forms, etc. This has been shown to happen in $N = 1$, $d = 4$ supergravity in ref. [12].

[3]As mentioned before the tensor hierarchy does not predict those potentials that cannot appear in the Stückelberg couplings. These tensors must be dealt with separately.

2 The $d = 5$ general tensor hierarchy

2.1 $d = 5$ bosonic field theories

In $d = 5$ dimensions vectors are dual to 2-forms. We can, therefore, use as a starting point, theories with spacetime metric $g_{\mu\nu}$, scalars ϕ^x parametrizing a target space with metric $g_{xy}(\phi)$ and 1-forms A^I only. The most general action with (ungauged and massless) Abelian gauge-invariance $\delta A^I = -d\Lambda^I$, no gauged symmetries and terms with no more than two derivatives that we can write for these fields is[4]

$$S = \int \left\{ \star R + \frac{1}{2} g_{xy}(\phi) d\phi^x \wedge \star d\phi^y - \frac{1}{2} a_{IJ}(\phi) F^I \wedge \star F^J - \star V(\phi) + \frac{1}{3} C_{IJK} F^I \wedge F^J \wedge A^K \right\}, \quad (2.1)$$

where

$$F^I = dA^I \,, \quad (2.2)$$

and where $g_{xy}(\phi)$ and $a_{IJ}(\phi)$ are symmetric, positive-definite matrices that depend on the scalar fields, $V(\phi)$ is a scalar potential and C_{IJK} is a constant, totally symmetric, tensor; any other components of C_{IJK} apart from the totally symmetric ones would not contribute to the action and, therefore, without loss of generality, they are set equal to zero.

This action takes exactly the same form as the bosonic action of minimal $d = 5$ supergravity coupled to vector supermultiplets and hypermultiplets (if we assume all the corresponding scalars are represented by the ϕ^x) given in ref. [17]. However, although probably most interesting applications of this work will be in the context of supergravity theories, we stress that here we are considering a general field theory in which there is no underlying real special geometry, the objects $g_{xy}(\phi)$, $a_{IJ}(\phi)$, and C_{IJK} need not be related by real special geometry as in the supersymmetric case and the scalars parametrize arbitrary target spaces and occur in a number which is unrelated to the number of vector fields.

From this point of view, the tensor C_{IJK} is just a set of possible deformations of the minimally coupled theory (which has $C_{IJK} = 0$). It gives rise to vector couplings unrelated to Yang-Mills gauge symmetry. This type of couplings are not possible in $d = 4$ dimensions.

If we only vary the 1-forms in the action, we get

$$\delta S = \int \left\{ -\delta A^I \wedge \star \frac{\delta S}{\delta A^I} \right\} , \qquad \star \frac{\delta S}{\delta A^I} = d(a_{IJ} \star F^J) - C_{IJK} F^J \wedge F^K \,, \quad (2.3)$$

and, on account of eq. (2.2), the equation of motion can be rewritten in the form

$$d(a_{IJ} \star F^J - C_{IJK} F^J \wedge A^K) = 0 \,. \quad (2.4)$$

This suggests to define the 2-forms B_I dual to the 1-forms A^I via

$$a_{IJ} \star F^J - C_{IJK} F^J \wedge A^K \equiv dB_I \,. \quad (2.5)$$

Since, by definition, $a_{IJ} \star F^J$ is gauge-invariant, the gauge-invariant field strengths of the 2-forms can be defined by

$$H_I \equiv dB_I + C_{IJK} A^J \wedge dA^K \,, \quad (2.6)$$

[4] Our conventions for differential forms, Hodge duals etc. can be found in appendix A.

so that we have the Bianchi identity and duality relation

$$dH_I = C_{IJK} F^J \wedge F^K\,, \qquad H_I = a_{IJ} \star F^J\,. \tag{2.7}$$

The gauge transformations of the 1- and 2-forms can be inferred from the gauge-invariance of their field strengths:

$$\delta_\Lambda A^I = -d\Lambda^I\,, \tag{2.8}$$

$$\delta_\Lambda B_I = d\Lambda_I + C_{IJK}\Lambda^J F^K\,. \tag{2.9}$$

The construction of the tensor hierarchy based on the embedding-tensor formalism should reproduce these results in the ungauged limit $\vartheta_I{}^A$ (with any possible other deformation tensor not being C_{IJK} sent to zero as well).

2.2 Gaugings and massive deformations

Let us consider the infinitesimal global transformations with constant parameters α^A of the scalars ϕ^x, 1-forms A^I and dual 2-forms B_I:

$$\delta_\alpha \phi^x = \alpha^A k_A{}^x(\phi)\,, \tag{2.10}$$

$$\delta_\alpha A^I = \alpha^A T_{A\,J}{}^I A^J\,, \tag{2.11}$$

$$\delta_\alpha B_I = -\alpha^A T_{A\,I}{}^J B_J\,, \tag{2.12}$$

where the matrices T_A belong to some representation of a group G and the $k_A{}^x(\phi)$ are the contravariant components of vectors defined on the scalar manifold. Some of the matrices and the vectors may be identically zero. They satisfy the algebras

$$[T_A, T_B] = -f_{AB}{}^C T_C\,, \qquad [k_A, k_B] = -f_{AB}{}^C k_C\,. \tag{2.13}$$

These transformations will be global symmetries of the theory constructed in the previous section if the following four conditions are met:

1. The vectors $k_A{}^x(\phi)$ are Killing vectors of the metric $g_{xy}(\phi)$ of the scalar manifold.
2. The kinetic matrix a_{IJ} satisfies the condition

$$\pounds_A a_{IJ} = -2T_{A\,(I}{}^K a_{J)K}\,, \tag{2.14}$$

   where $\pounds_A$ denotes the Lie derivative along the vector k_A.
3. The deformation tensor is invariant

$$\delta_A C_{IJK} \equiv Y_{A\,IJK} = -3T_{A\,(I}{}^L C_{JK)L} = 0\,. \tag{2.15}$$

4. The scalar potential is invariant

$$\pounds_A V = k_A V = 0\,. \tag{2.16}$$

In what follows, we will relax these conditions. Conditions 1 and 2 above cannot be relaxed but it is unnecessarily restrictive to demand that the symmetry group of the minimally coupled undeformed theory which has $C_{IJK} = 0$ and $V = 0$ is equal to the symmetry group G. More generally we can allow $\delta_A C_{IJK} = Y_{A\,IJK} \neq 0$ and $\pounds_A V = k_A V \neq 0$ and instead consider that subgroup of G under which C_{IJK} and V are invariant. In this way we have the situation that C_{IJK} and V introduce deformations that break the symmetry group G of the undeformed theory to a subgroup of G.

From the point of view of the construction of gauge-invariant theories using the embedding tensor formalism the above conditions 3 and 4 are also unnecessary. In general, the embedding tensor projects the above transformations into a smaller subgroup of G. The theory that we will construct will be only required to be invariant under gauge transformations of this smaller subgroup, but not necessarily under all the above global transformations. In the ungauged limit, i.e. setting the embedding tensor equal to zero, the theory will be invariant under the global transformations of the gauge group and not necessarily under any other global transformations.

From the general construction of the de- and top-form potentials, explained in the introduction, we know that if the tensor C_{IJK} is invariant under the transformations generated by all the matrices T_A, then the tensor $Y_{A\,IJK}$ will vanish identically and there will not be a non-trivial 4-form potential D^{IJK} dual to C_{IJK}. There are cases of physical interest (such as the maximal $d = 5$ supergravity of ref. [15]) in which this is what happens.

After these comments, we can now proceed to gauge the above transformations. This can be done by promoting the constant parameters α^A to arbitrary functions and using the 1-forms as gauge fields. The embedding tensor $\vartheta_I{}^A$ will relate the symmetry to be gauged with the 1-form that will gauge it:

$$\alpha^A(x) \equiv \Lambda^I \vartheta_I{}^A \,. \tag{2.17}$$

Thus, we want the theory to be invariant under the local transformations of the scalars

$$\delta_\Lambda \phi^x = \Lambda^I \vartheta_I{}^A k_A{}^x(\phi) \,, \tag{2.18}$$

and for this we need the covariant derivatives

$$\mathfrak{D}\phi^x \equiv d\phi^x + A^I \vartheta_I{}^A k_A{}^x(\phi) \,. \tag{2.19}$$

It can be checked that $\mathfrak{D}\phi^x$ transforms covariantly if we impose the quadratic constraint

$$Q_{IJ}{}^A \equiv -\delta_I \vartheta_J{}^A = \vartheta_I{}^B T_{B\,J}{}^K \vartheta_K{}^A - \vartheta_I{}^B \vartheta_J{}^C f_{BC}{}^A = 0 \,, \tag{2.20}$$

and impose that the vectors transform according to

$$\delta_\Lambda A^I = -\mathfrak{D}\Lambda^I + \Delta A^I = -\left(d\Lambda^I + \vartheta_J{}^A T_{A\,K}{}^I A^J \Lambda^K\right) + \Delta A^I \,, \qquad \vartheta_I{}^A \Delta A^I = 0 \,, \tag{2.21}$$

where the term ΔA^I is, otherwise and so far, arbitrary.

The above quadratic constraint means that $\vartheta_I{}^A$ is an invariant tensor since

$$\delta_\Lambda \vartheta_I{}^A = -\Lambda^J Q_{JI}{}^A = \Lambda^J \vartheta_J{}^B Y_{B\,I}{}^A = 0 \,, \tag{2.22}$$

where

$$Y_{A\,I}{}^{B} \equiv \delta_A \vartheta_I{}^B = \vartheta_I{}^C f_{AC}{}^B - T_{A\,I}{}^K \vartheta_K{}^B\,, \tag{2.23}$$

is the Y-tensor associated to the quadratic constraint according to the general formalism explained in the introduction.

2.2.1 The 2-form field strengths F^I

The next step is to construct the field strength F^I of the 1-forms. If we take the covariant derivative of the scalars' covariant "field strength" $\mathfrak{D}\phi^x$ we find

$$\mathfrak{D}\mathfrak{D}\phi^x = \left(dA^I + \frac{1}{2} X_{JK}{}^I A^{JK}\right) \vartheta_I{}^A k_A{}^x\,, \tag{2.24}$$

where, from now on, we use the shorthand notation[5]

$$A^{I\cdots J} \equiv A^I \wedge \cdots \wedge A^J\,, \qquad dA^{I\cdots J} \equiv dA^I \wedge \cdots \wedge dA^J\,, \qquad F^{I\cdots J} \equiv F^I \wedge \cdots \wedge F^J\,, \quad \text{etc.} \tag{2.25}$$

and where we have defined, as is customary, the X generators

$$X_{IJ}{}^K \equiv \vartheta_I{}^A T_{A\,J}{}^K\,. \tag{2.26}$$

Since the left hand side of the above Bianchi identity is covariant, by construction, the right hand side is also covariant and it is natural[6] to define

$$\mathfrak{D}\mathfrak{D}\phi^x = F^I \vartheta_I{}^A k_A{}^x\,, \tag{2.27}$$

$$F^I \equiv dA^I + \frac{1}{2} X_{JK}{}^I A^{JK} + \Delta F^I\,, \tag{2.28}$$

$$\vartheta_I{}^A \Delta F^I = 0\,. \tag{2.29}$$

Requiring gauge-covariance of F^I one finds that the term ΔF^I must transform according to

$$\delta_\Lambda \Delta F^I = -\mathfrak{D}\Delta A^I + 2X_{(JK)}{}^I \left[\Lambda^J F^K + \frac{1}{2} A^J \wedge \delta_\Lambda A^K\right]. \tag{2.30}$$

In order to satisfy the constraint $\vartheta_I{}^A \Delta F^I = \vartheta_I{}^A \Delta A^I = 0$ we introduce a Stückelberg tensor Z^{IJ} satisfying

$$Q^{AI} \equiv \vartheta_J{}^A Z^{JI} = 0\,, \tag{2.31}$$

and define

$$\Delta F^I \equiv Z^{IJ} B_J\,, \qquad \Delta A^I \equiv -Z^{IJ} \Lambda_J\,, \tag{2.32}$$

where Λ_I are the 1-form gauge parameters under which the 2-forms B_I must transform.

Observe that the constraint (2.31) tells us that the 2-forms can only occur as Stückelberg fields in the ungauged vector field strengths. Only the ungauged vector fields can be eaten up by the 2-forms which will become massive. We are thus describing through

[5] We will use a similar notation for exterior products of 2-forms and 3-forms throughout the rest of the paper, for example: $B_{IJ} \equiv B_I \wedge B_J$ etc.

[6] Actually, it can be argued that this is the only solution that does not require the introduction of additional fields in the theory.

the introduction of Z^{IJ} besides gaugings also massive deformations of the theory described in section 2.1.

The gauge transformation of ΔF^I implies

$$Z^{IJ}\delta_\Lambda B_J = Z^{IJ}\mathfrak{D}\Lambda_J + 2X_{(JK)}{}^I\left[\Lambda^J F^K + \frac{1}{2}A^J \wedge \delta_\Lambda A^K\right]. \tag{2.33}$$

This solution will only work if $X_{(JK)}{}^I \sim Z^{IL}\mathcal{O}_{JKL}$ for some tensor $\mathcal{O}_{JKL}$ symmetric, at least, in the last two indices. It is natural to identify this tensor with the fully symmetric tensor C_{IJK} that we know can occur in a Chern-Simons term in the action. This identification allows us to recover the theory of section 2.1 in the $\vartheta_I{}^A, Z^{IJ} \to 0$ limit.

Thus, we impose the constraint[7]

$$Q_{JK}{}^I \equiv X_{(JK)}{}^I - Z^{IL}C_{JKL} = 0\,, \tag{2.34}$$

and find that the field strength

$$F^I = dA^I + \frac{1}{2}X_{JK}{}^I A^{JK} + Z^{IJ}B_J\,, \tag{2.35}$$

transforms gauge-covariantly under the gauge transformations:

$$\delta_\Lambda A^I = -\mathfrak{D}\Lambda^I - Z^{IJ}\Lambda_J\,, \tag{2.36}$$

$$\delta_\Lambda B_J = \mathfrak{D}\Lambda_J + 2C_{JKL}\left(\Lambda^K F^L + \frac{1}{2}A^K \wedge \delta_\Lambda A^L\right) + \Delta B_J\,, \qquad Z^{IJ}\Delta B_J = 0\,, \tag{2.37}$$

where the possible additional term ΔB_J will be determined by the requirement of gauge-covariance of the 3-form field strength H_J.

The Stückelberg tensor Z^{IJ} and the Chern-Simons tensor C_{IJK} have to be gauge-invariant tensors, which, following the convention in eq. (1.2), leads to the constraints

$$Q_L{}^{IJ} \equiv \delta_L Z^{IJ} = -\left(X_{LK}{}^I Z^{KJ} + X_{LK}{}^J Z^{IK}\right) = 0\,, \tag{2.38}$$

$$Q_{IJKL} \equiv \delta_I C_{JKL} = 3X_{I(J}{}^M C_{KL)M} = 0\,, \tag{2.39}$$

and to the Y-tensors

$$Y_A{}^{IJ} \equiv \delta_A Z^{IJ} = T_{A\,K}{}^I Z^{KJ} + T_{A\,K}{}^J Z^{IK}\,, \tag{2.40}$$

and $Y_{A\,IJK}$ given in eq. (2.15), which are both annihilated by the embedding tensor by virtue of the above constraints.

2.2.2 The 3-form field strengths H_I

The covariant derivative of the 2-form field strength F^I, after use of the generalized Jacobi identities

$$X_{[JK}{}^M X_{L]M}{}^I = \frac{2}{3}Z^{IN}X_{[JK}{}^M C_{L]MN}\,, \tag{2.41}$$

[7] In $d = 4$ dimensions there is a similar constraint which is linear in the embedding tensor. In $d = 5$ the constraint has terms linear and of zeroth order in the embedding tensor.

is

$$\mathfrak{D}F^I = Z^{IJ}\left[\mathfrak{D}B_J + C_{JKL}A^K \wedge dA^L + \frac{1}{3}C_{JP[K}X_{ML]}{}^P A^{KML}\right], \tag{2.42}$$

which leads us to define the 3-form field strength

$$\mathfrak{D}F^I = Z^{IJ}H_J\,, \tag{2.43}$$

$$H_J \equiv \mathfrak{D}B_J + C_{JKL}A^K \wedge dA^L + \frac{1}{3}C_{JP[K}X_{ML]}{}^P A^{KML} + \Delta H_J\,, \tag{2.44}$$

$$Z^{IJ}\Delta H_J = 0\,, \tag{2.45}$$

where ΔH_J will be determined, together with ΔB_J by requiring gauge-covariance of H_J. Instead of constructing gauge transformations realizing gauge-covariance we construct a Bianchi identity for H_I in terms of gauge-covariant objects.

Let us first take the covariant derivative of both sides of the Bianchi identity of F^I eq. (2.43). Using the Ricci identity

$$\mathfrak{D}\mathfrak{D}F^I = X_{JK}{}^I F^{JK} = Z^{IL}C_{LJK}F^{JK}\,, \tag{2.46}$$

we find

$$Z^{IL}(\mathfrak{D}H_L - C_{LJK}F^{JK}) = 0\,, \tag{2.47}$$

which implies that the Bianchi identity for H_I must have the form[8]

$$\mathfrak{D}H_I = C_{IJK}F^{JK} + \Delta\mathfrak{D}H_I\,, \qquad Z^{JI}\Delta\mathfrak{D}H_I = 0\,, \tag{2.48}$$

which, in turn, implies that $\Delta\mathfrak{D}H_I$ must be proportional to the invariant tensor(s) we mentioned before. To find them, we have to compute directly $\mathfrak{D}H_I$ using the above expression.

In order to make progress in the calculation we must impose the constraint

$$Z^{IJ} = -Z^{JI}\,. \tag{2.49}$$

This property implies that the quadratic constraint $Q_I{}^{JK}$ and tensor $Y_A{}^{JK}$ can be written in the form

$$Q_I{}^{JK} = 2X_{IL}{}^{[J}Z^{K]L}\,, \qquad Y_A{}^{JK} = -2T_{A\,L}{}^{[J}Z^{K]L}\,. \tag{2.50}$$

A tensor with properties similar to those of Z^{IJ} appears in $N = 2$, $d = 5$ supergravity with general couplings to vector and tensor supermultiplets in ref. [17].

2.2.3 The 4-form field strengths G_A

Using eqs. (2.31) and (2.49) we find that ΔH_I and $\Delta\mathfrak{D}H_I$ can be taken to be

$$\Delta H_I = \vartheta_I{}^A C_A\,, \qquad \Delta\mathfrak{D}H_I = \vartheta_I{}^A G_A\,, \tag{2.51}$$

where $\vartheta_I{}^A G_A$ is the gauge-covariant field strength of the 3-forms $\vartheta_I{}^A C_A$. This determines the Bianchi identity of H_I to be

$$\mathfrak{D}H_I = C_{IJK}F^{JK} + \vartheta_I{}^A G_A\,. \tag{2.52}$$

[8] $\Delta\mathfrak{D}H_I$ should not be confused with $\mathfrak{D}\Delta H_I$.

An explicit computation of $\mathfrak{D}H_I$ gives

$$G_A = \mathfrak{D}C_A + T_{A\,K}{}^I \left[\left(F^K - \frac{1}{2} Z^{KL} B_L \right) \wedge B_I + \frac{1}{3} C_{ILM} A^{KL} \wedge dA^M + \frac{1}{12} C_{ILP} X_{MN}{}^P A^{KLMN} \right] + \Delta G_A \,, \tag{2.53}$$

$$\vartheta_I{}^A \Delta G_A = 0 \,. \tag{2.54}$$

According to the general scheme outlined in the introduction we expect that ΔG_A will be formed out of terms proportional to the three Y-tensors $Y_{A\,I}{}^B = \delta_A \vartheta_I{}^B$, $Y_A{}^{IJ} = \delta_A Z^{IJ}$, $Y_{A\,IJK} = \delta_A C_{IJK}$ associated to the three deformation tensors, contracted with some de-form potentials. Each of these Y-tensors is annihilated by the embedding tensor. We will next confirm that this is indeed what happens.

2.2.4 The 5-form field strengths K

To find the invariant tensors and de-forms that make up ΔG_A we follow the same procedure as before and take the covariant derivative of both sides of the Bianchi identity (2.52) for H_I. Using the Ricci identity

$$\mathfrak{D}\mathfrak{D}H_I = -\vartheta_J{}^A T_{A\,I}{}^K F^J \wedge H_K \,, \tag{2.55}$$

and the Bianchi identities for F^I and H_I, we get

$$\vartheta_I{}^A \left[\mathfrak{D}G_A - T_{A\,J}{}^K F^J \wedge H_K \right] = 0 \,, \tag{2.56}$$

from which it follows that the Bianchi identity for G_A will have the form

$$\mathfrak{D}G_A = T_{A\,J}{}^K F^J \wedge H_K + \Delta\mathfrak{D}G_A \,, \qquad \vartheta_I{}^A \Delta\mathfrak{D}G_A = 0 \,. \tag{2.57}$$

This implies that $\Delta\mathfrak{D}G_A$ must be proportional to the same invariant tensors that ΔG_A is proportional to. A direct calculation of $\mathfrak{D}G_A$ gives the result

$$\begin{aligned} \mathfrak{D}G_A = {} & T_{A\,K}{}^I F^K \wedge H_I \\ & + Y_A{}^{IJ} \left[\frac{1}{2} \mathfrak{D}B_I - H_I \right] \wedge B_J \\ & + Y_{A\,I}{}^B \left[(F^I - Z^{IL} B_L) \wedge C_B + \frac{1}{12} T_{B\,J}{}^M C_{KML} A^{IJK} \wedge dA^L \right. \\ & \qquad \left. + \frac{1}{60} T_{B\,J}{}^N C_{KPN} X_{LM}{}^P A^{IJKLM} \right] \\ & + Y_{A\,IJK} \left[\frac{1}{3} A^I \wedge dA^{JK} + \frac{1}{4} X_{LM}{}^K A^{ILM} \wedge dA^J + \frac{1}{20} X_{LM}{}^J X_{NP}{}^K A^{ILMNP} \right] \\ & + \mathfrak{D}\Delta G_A \,. \end{aligned} \tag{2.58}$$

This tells us that we must introduce three de-forms D^{IJ}, $D^I{}_A$ and D^{IJK}, with the same symmetries as the respective Y-tensors, and take

$$\Delta G_A = Y_A{}^{IJ} D_{IJ} + Y_{A\,I}{}^B D^I{}_B + Y_{A\,IJK} D^{IJK} \,, \tag{2.59}$$

in order for $\mathfrak{D}G_A$ to be gauge-covariant. This is simply because the terms proportional to the Y-tensors must each be gauge-covariant and this can only be the case of they form field strengths of de-forms. The ad-form field strength G_A and its Bianchi identity take the final form

$$\begin{aligned} G_A &= \mathfrak{D}C_A + T_{AK}{}^I\left[\left(F^K - \frac{1}{2}Z^{KL}B_L\right)\wedge B_I + \frac{1}{3}C_{ILM}A^{KL}\wedge dA^M + \frac{1}{12}C_{ILP}X_{MN}{}^PA^{KLMN}\right] \\ &\quad + Y_A{}^{IJ}D_{IJ} + Y_{AI}{}^BD^I{}_B + Y_{AIJK}D^{IJK}\,, \qquad (2.60) \\ \mathfrak{D}G_A &= T_{AK}{}^IF^K\wedge H_I + Y_A{}^{IJ}K_{IJ} + Y_{AI}{}^BK^I{}_B + Y_{AIJK}K^{IJK}\,, \qquad (2.61) \end{aligned}$$

where

$$\begin{aligned} K_{IJ} &\equiv \mathfrak{D}D_{IJ} - \left[H_{[I} - \frac{1}{2}\mathfrak{D}B_{[I}\right]\wedge B_{J]} + \Delta K_{IJ}\,, \qquad (2.62) \\ K^I{}_B &\equiv \mathfrak{D}D^I{}_B + (F^I - Z^{IL}B_L)\wedge C_B + \frac{1}{12}T_{BJ}{}^MC_{KML}A^{IJK}\wedge dA^L \\ &\quad + \frac{1}{60}T_{BJ}{}^NC_{KPN}X_{LM}{}^PA^{IJKLM} + \Delta K^I{}_B\,, \qquad (2.63) \\ K^{IJK} &\equiv \mathfrak{D}D^{IJK} + \frac{1}{3}A^{(I}\wedge dA^{JK)} + \frac{1}{4}X_{LM}{}^{(K}A^{I|LM}\wedge dA^{|J)} + \frac{1}{20}X_{LM}{}^{(J}X_{NP}{}^KA^{I)LMNP} \\ &\quad + \Delta K^{IJK}\,, \qquad (2.64) \end{aligned}$$

in which ΔK_{IJ}, $\Delta K^I{}_B$ and ΔK^{IJK} satisfy

$$Y_A{}^{IJ}\Delta K_{IJ} + Y_{AI}{}^B\Delta K^I{}_B + Y_{AIJK}\Delta K^{IJK} = 0\,. \qquad (2.65)$$

As explained in the introduction the terms ΔK will be contractions of (W-)tensors and 5-form potentials. To determine the W-tensors and the 5-form potentials, we take the covariant derivative of the Bianchi identity of G_A, eq. (2.61). Ignoring the fact that we are working in $d = 5$ dimensions we get

$$Y_A{}^{IJ}\left[\mathfrak{D}K_{IJ} - \frac{1}{2}H_{IJ}\right] + Y_{AI}{}^B\left[\mathfrak{D}K^I{}_B - F^I\wedge G_B\right] + Y_{AIJK}\left[\mathfrak{D}K^{IJK} - \frac{1}{3}F^{IJK}\right] = 0\,. \qquad (2.66)$$

If we take the covariant derivative of the above expression, we find

$$\begin{aligned} F^K\wedge K_{MN}\{+2Y_A{}^{IM}X_{KI}{}^N - Y_{AK}{}^BY_B{}^{MN}\} \\ +F^{KL}\wedge H_M\{-Y_A{}^{IM}C_{KLI} - Y_{AL}{}^BT_{BK}{}^M - Y_{AIKL}Z^{IM}\} \\ +G_B\wedge H_J\{-Y_A{}^{IJ}\vartheta_I{}^B - Y_{AI}{}^BZ^{IJ}\} \\ +F^I\wedge K^{JKL}\{-Y_{AI}{}^BY_{BJKL} + 3Y_{AMJK}X_{IL}{}^M\} \\ +F^K\wedge K^J{}_D\{Y_{AI}{}^BW_B{}^I{}_{KJ}{}^D\} = 0\,, \qquad (2.67) \end{aligned}$$

where

$$W_B{}^I{}_{KJ}{}^D \equiv \vartheta_K{}^Cf_{BC}{}^D\delta_J{}^I + X_{KJ}{}^I\delta_B{}^D - Y_{BJ}{}^D\delta_K{}^I\,, \qquad (2.68)$$

as in $d = 4$.

Each term in braces is linear (or quadratic) in Y-tensors and vanishes identically upon use of the 5 constraints $Q_I{}^{JK}, Q_{IJ}{}^K, Q^{AI}, Q_{IJKL}, Q_{IJ}{}^A$. Furthermore, the index structure of the products of field strengths which multiply the 5 expressions in braces coincides with that of the duals of those 5 constraints. Actually, each of those terms corresponds to one of the identities in eq. (1.10), and we can rewrite the above expression in the form

$$
\begin{aligned}
F^{I}\wedge K_{JK}\left\{Y_{A}{}^{LM}\frac{\partial Q_{I}{}^{JK}}{\partial Z^{LM}}+Y_{A\,L}{}^{B}\frac{\partial Q_{I}{}^{JK}}{\partial\vartheta_{L}{}^{B}}\right\}&\\
+F^{IJ}\wedge H_{K}\left\{Y_{A}{}^{LM}\frac{\partial Q_{IJ}{}^{K}}{\partial Z^{LM}}+Y_{A\,L}{}^{B}\frac{\partial Q_{IJ}{}^{K}}{\partial\vartheta_{L}{}^{B}}+Y_{A\,LMN}\frac{\partial Q_{IJ}{}^{K}}{\partial C_{LMN}}\right\}&\\
+G_{B}\wedge H_{I}\left\{Y_{A}{}^{JK}\frac{\partial Q^{B\,I}}{\partial Z^{JK}}+Y_{A\,J}{}^{C}\frac{\partial Q^{B\,I}}{\partial\vartheta_{J}{}^{C}}\right\}&\\
+F^{I}\wedge K^{JKL}\left\{Y_{A\,M}{}^{B}\frac{\partial Q_{I\,JKL}}{\partial\vartheta_{M}{}^{B}}+3Y_{A\,MNP}\frac{\partial Q_{I\,JKL}}{\partial C_{MNP}}\right\}&\\
+F^{I}\wedge K^{J}{}_{B}\left\{Y_{A\,K}{}^{C}\frac{\partial Q_{IJ}{}^{B}}{\partial\vartheta_{K}{}^{C}}\right\}&=0\,.
\end{aligned}
\tag{2.69}
$$

The scheme explained in the introduction leads us to assume the existence of five 5-forms $E^{I}{}_{JK}, E^{IJ}{}_{K}, E_{A\,I}, E^{I\,JKL}, E^{IJ}{}_{A}$ dual to the 5 constraints $Q_{I}{}^{JK}$, $Q_{IJ}{}^{K}, Q^{A\,I}, Q_{I\,JKL}, Q_{IJ}{}^{A}$ so

$$
\Delta K_{IJ} \equiv +2X_{K[I}{}^{L}E^{K}{}_{J]L}-C_{KL[I}E_{J]}{}^{KL}-\vartheta_{[I|}{}^{A}E_{A\,|J]}\,, \tag{2.70}
$$

$$
\begin{aligned}
\Delta K_{B}{}^{I} \equiv\ & W_{B}{}^{I}{}_{KJ}{}^{D}E^{KJ}{}_{D}-Z^{IJ}E_{B\,J}-T_{B\,K}{}^{J}E_{J}{}^{IK}-Y_{B}{}^{JK}E^{I}{}_{JK}\\
&-Y_{B\,JKM}E^{I\,JKM}\,,
\end{aligned}
\tag{2.71}
$$

$$
\Delta K^{IJK} \equiv 3X_{LM}{}^{(I|}E^{L\,|JK)M}+Z^{L(I}E_{L}{}^{JK)}\,. \tag{2.72}
$$

Each of these expressions is of the form $\Delta K_{\sharp}=\sum_{\flat}E_{\flat}\partial Q^{\flat}/\partial c^{\sharp}$.

With the determination of the 5-form field strengths K we have completed the construction of the 5-dimensional tensor hierarchy. The gauge transformations of all the potentials can be obtained by constructing the most general gauge transformations under which all the field strengths transform gauge-covariantly. We will not proceed to determine these gauge transformations as they are in principle determined by the Bianchi identities.

2.2.5 Gauge-invariant action for the 1- and 2-forms

The gauge-invariant action for the 1- and 2-forms is essentially the one given in ref. [15], with the E_6 tensors Z^{IJ}, C_{IJK} replaced by arbitrary tensors satisfying the five algebraic constraints, giving:

$$
\begin{aligned}
S=\int\Big\{&\star R+\tfrac{1}{2}g_{xy}(\phi)\mathfrak{D}\phi^{x}\wedge\star\mathfrak{D}\phi^{y}-\tfrac{1}{2}a_{IJ}(\phi)F^{I}\wedge\star F^{J}-\star V(\phi)\\
&-Z^{IJ}B_{I}\wedge\Big[H_{J}-\tfrac{1}{2}\mathfrak{D}B_{J}\Big]+\tfrac{1}{3}C_{IJK}\Big[A^{I}\wedge dA^{JK}+\tfrac{3}{4}X_{LM}{}^{I}A^{JLM}\wedge dA^{K}\\
&+\tfrac{3}{20}X_{LM}{}^{I}X_{NP}{}^{J}A^{LMNPK}\Big]\Big\}\,,
\end{aligned}
\tag{2.73}
$$

where the scalar potential $V(\phi)$ may contain more terms than the one in eq. (2.1). The new terms must depend on the deformation tensors in such a way that the potential of the ungauged theory is recovered when they are set to zero.

A general variation of the above action can be written in the form[9]

$$\delta S \equiv \int \left\{ \delta g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} - \delta\phi^x \star \frac{\delta S}{\delta \phi^x} - \delta A^I \wedge \star \frac{\widetilde{\delta S}}{\delta A^I} - (\delta B_I - C_{IJK} A^J \wedge \delta A^K) \wedge \star \frac{\delta S}{\delta B_I} \right\} , \quad (2.74)$$

where the equations of motion are[10]

$$\begin{aligned} \frac{\delta S}{\delta g^{\mu\nu}} &= \star \left\{ G_{\mu\nu} + \frac{1}{2} g_{xy} \left[\mathfrak{D}_\mu \phi^x \mathfrak{D}_\nu \phi^y - \frac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho \phi^x \mathfrak{D}^\rho \phi^y \right] \right. \\ &\qquad \left. - \frac{1}{2} a_{IJ} \left[F^I{}_\mu{}^\rho F^J{}_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^{I\,\rho\sigma} F^J{}_{\rho\sigma} \right] + \frac{1}{2} g_{\mu\nu} V \right\} , \end{aligned} \quad (2.76)$$

$$\star \frac{\delta S}{\delta \phi^x} = g_{xy} \mathfrak{D} \star \mathfrak{D} \phi^y + \frac{1}{2} \partial_x a_{IJ} F^I \wedge \star F^J + \star \partial_x V , \quad (2.77)$$

$$\star \frac{\widetilde{\delta S}}{\delta A^I} = \mathfrak{D}(a_{IJ} \star F^J) - C_{IJK} F^{JK} - \star \vartheta_I{}^A j_A , \quad (2.78)$$

$$\star \frac{\delta S}{\delta B_I} = -Z^{IJ}(a_{JK} \star F^K - H_J) , \quad (2.79)$$

in which we have defined the 1-form currents

$$j_A \equiv k_{A\,x} \mathfrak{D} \phi^x . \quad (2.80)$$

Now, we can substitute in the general variation of the action the gauge transformations of the fields

$$\delta_\Lambda \phi^x = \Lambda^I \vartheta_I{}^A k_A{}^x , \quad (2.81)$$

$$\delta_\Lambda A^I = -\mathfrak{D} \Lambda^I - Z^{IJ} \Lambda_J , \quad (2.82)$$

$$\delta_\Lambda B_I = \mathfrak{D} \Lambda_I + 2 C_{IJK} \left(\Lambda^J F^K + \frac{1}{2} A^J \wedge \delta_\Lambda A^K \right) . \quad (2.83)$$

Checking invariance of the action under the gauge transformations generated by 0- and 1-form parameters amounts to checking the following two Noether identities:

$$\mathfrak{D} \star \frac{\widetilde{\delta S}}{\delta A^I} + 2 C_{IJK} F^J \wedge \star \frac{\delta S}{\delta B_K} + \vartheta_I{}^A k_A{}^x \star \frac{\delta S}{\delta \phi^x} = 0 , \quad (2.84)$$

$$\mathfrak{D} \star \frac{\delta S}{\delta B_I} + Z^{IJ} \star \frac{\widetilde{\delta S}}{\delta A^J} = 0 . \quad (2.85)$$

The second identity is easily seen to be satisfied. The first identity can also be shown to be satisfied upon use of the Killing property of $\vartheta_I{}^A k_A{}^x$, the property

$$\vartheta_I{}^A k_A a_{JK} = -2 X_{I(J}{}^L a_{K)L} , \quad (2.86)$$

[9] The tilde in the first variation w.r.t. the 1-forms A^I defines a modified first variation which has a simpler form than the total first variation which would be, as usual, the sum of all the terms proportional to δA^I and contains terms proportional to the equations of motion of other fields. We will use similar simplified first variations in the 6-dimensional action.

[10] Explicitly, we have

$$\mathfrak{D} \star \mathfrak{D} \phi^x = d \star \mathfrak{D} \phi^x + \Gamma_{yz}{}^x \mathfrak{D} \phi^y \wedge \star \mathfrak{D} \phi^z + \vartheta_I{}^A \partial_y k_A{}^x A^I \wedge \star \mathfrak{D} \phi^y . \quad (2.75)$$

of the kinetic matrix, the condition

$$\vartheta_I{}^A k_A V = 0\,, \tag{2.87}$$

of the scalar potential and the constraint $Q_{I\,JKL} = 0$. Observe that these are the same conditions required by global invariance but projected with the embedding tensor, which means they are weaker conditions.

We can now relate the equations of motion derived from this action and the tensor hierarchy's Bianchi identities via the duality relations

$$a_{IJ} \star F^J = H_I\,, \tag{2.88}$$

$$\star j_A = G_A\,, \tag{2.89}$$

$$\star \frac{\partial V}{\partial c^\sharp} = K_\sharp\,. \tag{2.90}$$

With these duality relations, the 1-form equations of motion become the Bianchi identities for the hierarchy's 3-form field strengths H_I. The projected scalar equations of motion $k_A{}^x \star \frac{\delta S}{\delta \phi^x}$ become the Bianchi identity of the hierarchy's 4-form field strengths G_A. In order to show this one must use the Killing property of the $k_A{}^x$, eq. (2.14) for the kinetic matrix, and the following expression for $k_A V$

$$k_A V = \sum_\sharp Y_A{}^\sharp \frac{\partial V}{\partial c^\sharp}\,. \tag{2.91}$$

Now that we have completed the construction of the 5-dimensional tensor hierarchy and provided an interpretation of the various potentials we summarize these results in table 1. We will explain the meaning of the table by discussing in detail the case of the 2-forms. The other forms then go analogously.

We have seen 2-forms appearing in the field strengths of the 1-forms. These are ungauged 1-forms because the field strengths of the gauged 1-forms do not contain any 2-forms. These 2-forms are $Z^{IJ}B_J$. Their gauge transformations are of the form $Z^{IJ}\delta B_J = Z^{IJ}\mathfrak{D}\Lambda_J$ plus terms involving the 0-form gauge transformation parameter Λ^I, but not the 2-form gauge transformation parameter Λ_A. Therefore, all the gauge transformations that the $Z^{IJ}B_J$ have are massless gauge transformations. This is indicated in table 1 by the term "massless" in the column called "gauge transformations". Since the $Z^{IJ}B_J$ 2-forms appear in the field strength of the ungauged 1-forms they form Stückelberg pairs with these ungauged 1-forms. This is indicated in table 1 by "ungauged A^I" in the column "Stückelberg pair with". It is not possible to say, unless we explicitly know all the components of Z^{IJ} exactly which 2-form B_I forms a Stückelberg pair with which 1-form A^I. Further, we also indicated that the 2-forms $Z^{IJ}B_J$ whose field strengths are $Z^{IJ}H_J$ are dual to $Z^{IJ}a_{JK}F^K$ and that 2-forms with these gauge transformation properties can only exist whenever $Z^{IJ} \neq 0$. Besides the 2-forms $Z^{IJ}B_J$ there are also those which do not appear in the field strengths of the 1-forms. Such 2-forms fall into two categories depending on their gauge transformation properties. The first possibility is that their field strengths contain Stückelberg couplings to 3-forms. These exist for those I for which the Stückelberg coupling tensor $\vartheta_I{}^A \neq 0$ and they will have massive gauge transformations.These 2-forms cannot also belong the $Z^{IJ}B_J$ type discussed earlier. Finally it can also happen that there are I values for which the 2-forms are not forming any Stückelberg pair with either 1-forms or 3-forms. Such 2-forms occur for example in the theory in which there is no embedding tensor nor the Stückelberg tensor Z. More generally they can occur in the gauged theory but only for those I for which $\vartheta_I{}^A = Z^{IJ} = 0$. The other entries of table 1 should be read in an analogous fashion.

The 1-forms have been left out from the table since they behave the same in any tensor hierarchy in any dimension. There are always three types: 1). gauged 1-forms which always have massless gauge transformations and exist for all those A for which $\vartheta_I{}^A \neq 0$, 2). ungauged 1-forms with massive gauge transformations which exist for all those I for which $Z^{IJ} \neq 0$ and 3). ungauged 1-forms with massless gauge transformations which exist for all those I for which $\vartheta_I{}^A = Z^{IJ} = 0$.

We end the discussion of the 5-dimensional tensor hierarchy with some comments about possible redundancy of potentials. Potentials that have massive gauge transformations can be totally gauged away, but which particular potentials have a massive gauge transformation (i.e. which p-form potentials are Stückelberg fields for a $(p+1)$-form potential) depends on the Stückelberg tensors occurring in their field strengths, as shown in table 1. Using a massive gauge transformation with a p-form (local) parameter to eliminate a p-form Stückelberg potential partially fixes the standard (massless) gauge transformations of the associated $(p+1)$-form potentials, which become massive. The top-forms are special because they have massive gauge transformations but they are not Stückelberg fields for any higher-rank potential.

For the p-forms with $p = 1, 2, 3$ this would lead to a (partial) gauge fixing of the 2-, 3- and 4-form gauge transformations. When this is done one can for example eliminate some of the 3-forms C_A for certain values of A. In the case of the 4-forms it can happen, depending on the details, that an entire form $D_\sharp$ can be gauged away. The 4-form massive gauge transformations are of the form $\delta D_\sharp = -W_\sharp{}^\flat \Lambda_\flat$ where $\Lambda_\flat$ is the 5-form gauge transformation parameter, $\delta E_\flat = \mathfrak{D}\Lambda_\flat$. The massive gauge transformations of the 4-forms $\delta D_\sharp = -W_\sharp{}^\flat \Lambda_\flat$ can sometimes be used to eliminate entirely some of the 4-forms $D_\sharp$. This happens for example in gauged maximal supergravity where there is only one deformation tensor, the embedding tensor, and hence there is only one 4-form. Similar statements apply to the 5-forms $E_\flat$ that always come contracted with $W_\sharp{}^\flat$ and are thus determined up to massive gauge transformations of the type $\delta E_\flat = \Sigma_\flat$ with $W_\sharp{}^\flat \Sigma_\flat = 0$.

3 The $d = 6$ general tensor hierarchy

3.1 $d = 6$ bosonic field theories

In $d = 6$ dimensions we can have, apart from a spacetime metric and scalars ϕ^x, n_1 1-forms A^i and n_2 electric 2-forms B^Λ. The 1-forms A^i are dual to 3-forms C_i and the electric 2-forms B^Λ are dual to magnetic 2-forms B_Λ (we will study their definitions later). Furthermore, in $d = 6$ dimensions we can have real (anti-) self-dual 3-forms and, therefore, we can constrain the 2-forms to have (anti-) self-dual 3-form field strengths.

We will write down an action ignoring momentarily the (anti-) self-duality constraint and impose it on the equations of motion derived from that action, as it was done in

Potential	Gauge transformation	Interpretation (field strength dual to)	Stückelberg pair with	Existence
B_I	massive	$a_{IJ}F^J$	$\vartheta_I{}^A C_A$	$\forall I : \vartheta_I{}^A \neq 0$
$Z^{IJ}B_J$	massless	$Z^{IJ}a_{JK}F^K$	ungauged A^I	$\forall I : Z^{IJ} \neq 0$
B_I	massless	$a_{IJ}F^J$	none	$\forall I : \vartheta_I{}^A = Z^{IJ} = 0$
C_A	massive	current j_A of symmetry broken by V	$Y_A{}^\sharp D_\sharp$	$\forall A : Y_A{}^\sharp \neq 0$
$\vartheta_I{}^A C_A$	massless	current j_A of gauged symmetry	B_I	$\forall I : \vartheta_I{}^A \neq 0$
C_A	massless	current j_A of global symmetry	none	$\forall A : Y_A{}^\sharp = \vartheta_I{}^A = 0$
$D_\sharp$	massive	$\partial V/\partial c^\sharp$	$W_\sharp{}^\flat E_\flat$	$\forall \sharp : W_\sharp{}^\flat \neq 0$
$Y_A{}^\sharp D_\sharp$	massless	$Y_A{}^\sharp \partial V/\partial c^\sharp$	C_A	$\forall A : Y_A{}^\sharp \neq 0$
$D_\sharp$	massless	$\partial V/\partial c^\sharp$	none	$\forall \sharp : W_\sharp{}^\flat = Y_A{}^\sharp = 0$
$W_\sharp{}^\flat E_\flat$	massless	enforces constraints	$D_\sharp$	$\forall \sharp : W_\sharp{}^\flat \neq 0$

Table 1. All the $p \geq 2$ forms of the 5-dimensional tensor hierarchy, their Stückelberg properties and physical interpretation.

$N = 2B$, $d = 10$ supergravity in refs. [18, 19]. This can only be done consistently if the field strengths and action are such that the Bianchi identities transform into the equations of motion and viceversa under electric-magnetic duality transformations of the 2-forms. In particular, if the action has Chern-Simons terms of the form $H \wedge F \wedge A$ which give rise to terms proportional to $F \wedge F$ in the equations of motion of the 2-forms, the field strengths H must necessarily have terms of the form $F \wedge A$.

Taking into account, thus, the possibility of having (anti-) self-dual 2-forms, the most general action with (ungauged and massless) Abelian gauge-invariance, with no more than two derivatives that we can write for scalars, vectors and (electric) 2-forms is, in differential form language,[11]

$$S = \int \Big\{ - \star R + \frac{1}{2} g_{xy}(\phi) d\phi^x \wedge \star d\phi^y - \frac{1}{2} a_{ij}(\phi) F^i \wedge \star F^j \\ + \frac{1}{2} b_{\Lambda\Sigma}(\phi) H^\Lambda \wedge \star H^\Sigma + \frac{1}{2} c_{\Lambda\Sigma}(\phi) H^\Lambda \wedge H^\Sigma + \star V(\phi) + \varepsilon d_{\Lambda\, ij} H^\Lambda \wedge F^i \wedge A^j \Big\} . \tag{3.1}$$

In this expression, F^i and H^Λ are the 2- and 3-form field strengths, defined by

$$F^i \equiv dA^i \,, \tag{3.2}$$

$$H^\Lambda \equiv dB^\Lambda + d^\Lambda{}_{ij} A^i \wedge dA^j \,, \tag{3.3}$$

invariant under the Abelian gauge transformations

$$\delta A^i = -d\Lambda^i \,, \tag{3.4}$$

$$\delta B^\Lambda = d\Lambda^\Lambda + d^\Lambda{}_{ij} \Lambda^i dA^j \,. \tag{3.5}$$

[11] See appendix A.

The scalar-dependent kinetic matrices $g_{xy}(\phi), b_{\Lambda\Sigma}(\phi), a_{ij}(\phi)$ are symmetric. The first two of them are positive-definite and the third is negative-definite. The tensor $c_{\Lambda\Sigma}(\phi)$ is antisymmetric. The constant tensors $d_{\Lambda\, ij}$ and $d^{\Lambda}{}_{ij}$ have the symmetries[12]

$$d_{\Lambda\, ij} = d_{\Lambda\, ji}\,, \qquad d^{\Lambda}{}_{ij} = d^{\Lambda}{}_{ji}\,, \tag{3.6}$$

ans satisfy the constraint

$$d_{\Lambda\, i(j} d^{\Lambda}{}_{kl)} = 0\,, \tag{3.7}$$

for the last term in the action to be gauge-invariant. We will later choose the arbitrary constant ε to have simple duality rules for the 2-forms.

If we vary the 1-forms and 2-forms in the action, we get

$$\delta S = \int \left\{ -\delta A^i \wedge \star \frac{\widetilde{\delta S}}{\delta A^i} - (\delta B^\Lambda + d^{\Lambda}{}_{ij} A^i \wedge \delta A^j) \wedge \star \frac{\delta S}{\delta B^\Lambda} \right\}, \tag{3.8}$$

where

$$\begin{aligned} \star \frac{\widetilde{\delta S}}{\delta A^i} &= d\Big\{ a_{ij} \star F^j - 2 d^{\Lambda}{}_{ij} A^j \wedge [J_\Lambda + \varepsilon d_{\Lambda\, kl} A^k \wedge dA^l] \\ &\qquad -2\varepsilon d_{\Lambda\, ij} H^\Lambda \wedge A^j - \frac{2}{3} \varepsilon d_{\Lambda\, ij} d^{\Lambda}{}_{kl} A^{jk} \wedge dA^l \Big\}, \end{aligned} \tag{3.9}$$

$$\star \frac{\delta S}{\delta B^\Lambda} = d\{ J_\Lambda + \varepsilon d_{\Lambda\, ij} A^i \wedge dA^j \}\,, \tag{3.10}$$

where we have defined

$$J_\Lambda \equiv b_{\Lambda\Sigma} \star H^\Sigma + c_{\Lambda\Sigma} H^\Sigma\,, \tag{3.11}$$

and where we have used the Bianchi identities and the property eq. (3.7) in order to write the equations of motion of the vector fields as total derivatives.

3.1.1 The magnetic 2-forms B_Λ

The equations of motion of the 2-forms B^Λ suggest the definition of the magnetic 2-forms B_Λ through

$$dB_\Lambda \equiv J_\Lambda + \varepsilon d_{\Lambda\, ij} A^i \wedge dA^j\,. \tag{3.12}$$

Since J_Λ is gauge-invariant, we define the dual 3-form field strengths by

$$H_\Lambda \equiv J_\Lambda = dB_\Lambda - \varepsilon d_{\Lambda\, ij} A^i \wedge dA^j\,. \tag{3.13}$$

We set $\varepsilon = -1$ to make the magnetic and electric 3-form field strengths as similar as possible. Thus, we can replace the equations of motion of the electric 2-forms, via the above definition of the magnetic field strengths, by a Bianchi identity.

[12]The Chern-Simons term containing $d_{\Lambda\, ij}$ in the Lagrangian is clearly symmetric in ij up to total derivatives. The terms containing $d^{\Lambda}{}_{ij}$, which appear in the field strengths H^Λ are symmetric up to a field redefinition of B^Λ.

In $d=6$ dimensions it is possible to constrain the 2-forms to have self- or anti-self-dual field strengths. We can write these constraints in the form

$$\zeta_{\Lambda\Omega}(H^{\Omega}-\zeta^{\Omega\Sigma}J_{\Sigma})=0\,, \tag{3.14}$$

where $\zeta^{\Lambda\Sigma}=\zeta_{\Lambda\Sigma}$ is a diagonal matrix whose diagonal components can only be $+1$ for self-dual 3-form field strengths, -1 for anti-self-dual 3-form field strengths or 0 for unconstrained 3-form field strengths. The (anti-)self-duality constraints will be consistent if the Bianchi identity for H^{Λ} becomes the equation of motion of B^{Λ} upon their use. The Bianchi identities and the equations of motion are

$$dH^{\Lambda} = d^{\Lambda}{}_{ij}F^{i}\wedge F^{j}\,, \tag{3.15}$$

$$dJ_{\Lambda} = d_{\Lambda\,ij}F^{i}\wedge F^{j}\,. \tag{3.16}$$

By hitting eq. (3.14) with an exterior derivative we find that the tensors $d^{\Lambda}{}_{ij}$, and $d_{\Lambda\,ij}$ must satisfy the constraint

$$\zeta_{\Omega\Lambda}(d^{\Lambda}{}_{ij}-\zeta^{\Lambda\Sigma}d_{\Sigma\,ij})=0\,, \tag{3.17}$$

for consistency.

3.1.2 The 3-forms C_i

The form of the equations of motion of the 1-forms also suggests the definition

$$\begin{aligned} dC_i \equiv\ & a_{ij}\star F^{j}-2d^{\Lambda}{}_{ij}A^{j}\wedge[J_{\Lambda}-d_{\Lambda\,kl}A^{k}\wedge dA^{l}]+2d_{\Lambda\,ij}H^{\Lambda}\wedge A^{j}\\ &+\frac{2}{3}d_{\Lambda\,ij}d^{\Lambda}{}_{kl}A^{jk}\wedge dA^{l}\,, \end{aligned} \tag{3.18}$$

or, using the magnetic 2-forms and the constraint eq. (3.7)

$$dC_i = a_{ij}\star F^{j}-2d^{M}{}_{ij}\left[A^{j}\wedge dB_{M}+\frac{1}{3}d_{M\,kl}A^{jk}\wedge dA^{l}\right]\,, \tag{3.19}$$

where we have defined the $2n_2$-component vectors

$$(B^{M})\equiv\begin{pmatrix}B^{\Lambda}\\ \\ B_{\Lambda}\end{pmatrix}\,,\qquad (d^{M}{}_{ij})\equiv\begin{pmatrix}d^{\Lambda}{}_{ij}\\ d_{\Lambda\,ij}\end{pmatrix}\,,\qquad (d_{M\,ij})\equiv\left(d_{\Lambda\,ij}\,,\ d^{\Lambda}{}_{ij}\right)\,. \tag{3.20}$$

The gauge-invariant 4-form field strengths G_i can be defined as

$$G_i\equiv dC_i+2d_{M\,ij}\left[A^{j}\wedge dB^{M}+\frac{1}{3}d^{M}{}_{kl}A^{jk}\wedge dA^{l}\right]\,, \tag{3.21}$$

which is related to the 2-form field strengths by the duality relation

$$G_i=a_{ij}\star F^{j}\,. \tag{3.22}$$

The 3-forms C_i can be redefined in order to make contact with the 3-forms that appear naturally in the tensor hierarchy. The redefinition is

$$C_i^{\rm old}\longrightarrow C_i^{\rm new}+2d_{M\,ij}B^{M}\wedge A^{j}\,, \tag{3.23}$$

so that

$$G_i = dC_i^{\rm new} + 2d^M{}_{ij}\left[dA^j \wedge B_M + \frac{1}{3}d_{M\,kl}A^{jk}\wedge dA^l\right]. \tag{3.24}$$

The Bianchi identity satisfied by G_i is

$$dG_i = 2d^M{}_{ij}F^j \wedge H_M\,. \tag{3.25}$$

In order to derive this it is useful to note that eq. (3.7) can also be written as

$$d_{M\,i(j}d^M{}_{kl)} = 0\,. \tag{3.26}$$

3.1.3 Symmetries

Let us momentarily set the d- and ζ-tensors to zero and consider the symmetries of the system of equations of motion and Bianchi identities of the 2-forms:

$$dH^\Lambda = 0\,, \tag{3.27}$$

$$dJ_\Lambda = 0\,. \tag{3.28}$$

This system is formally invariant under the $GL(2n_2,\mathbb{R})$ transformations

$$J^{M\prime} = M_N{}^M J^N\,, \qquad (J^M) \equiv \begin{pmatrix} H^\Lambda \\ \\ J_\Lambda \end{pmatrix}. \tag{3.29}$$

These transformations must be consistent with the definition of J_Λ in terms of H^Λ. Writing

$$(M_N{}^M) \equiv \begin{pmatrix} A_\Sigma{}^\Lambda & B^{\Sigma\Lambda} \\ C_{\Sigma\Lambda} & D^\Sigma{}_\Lambda \end{pmatrix}, \tag{3.30}$$

we find that, for consistency, the symmetric and antisymmetric kinetic matrices $b_{\Lambda\Sigma}, c_{\Lambda\Sigma}$ must transform according to

$$f' = (C + Df)(A + Bf)^{-1}\,, \tag{3.31}$$

$$f^{T\prime} = -(C - Df^T)(A - Bf^T)^{-1}\,, \tag{3.32}$$

where we have defined the matrix

$$f_{\Lambda\Sigma} = b_{\Lambda\Sigma} + c_{\Lambda\Sigma}\,. \tag{3.33}$$

Consistency between the two transformation rules implies

$$A^TC + C^TA = 0\,, \qquad B^TD + D^TB = 0\,, \qquad A^TD + C^TB = \xi\mathbb{I}_{n_2\times n_2}\,. \tag{3.34}$$

The constant ξ has to be $+1$ in order to preserve the energy-momentum tensor. The same conditions can be derived from the requirement that the matrix $M_N{}^M$ preserves the off-diagonal metric $(\eta^{MN}) = \begin{pmatrix} 0 & \mathbb{I}_{n_2\times n_2} \\ \mathbb{I}_{n_2\times n_2} & 0 \end{pmatrix}$, that is

$$M_M{}^P\eta_{PQ}M_N{}^Q = \eta_{MN}\,. \tag{3.35}$$

Thus, the system of 2-form equations of motion and Bianchi identities is invariant under symmetries that can be embedded into $\mathrm{SO}(n_2, n_2)$. The off-diagonal metric η can be used to raise and lower $M, N = 1, \cdots, 2n_2$ indices, in agreement with the definitions (3.20) of the vectors $d^M{}_{ij}$ and $d_{M\,ij}$.

Only those transformations of the matrices $b_{\Lambda\Sigma}$ and $c_{\Lambda\Sigma}$ that can be compensated by a reparametrization of the scalar manifold leaving invariant the target-space metric $g_{xy}(\phi)$ will be symmetries of the theory. Furthermore, the reparametrizations of the scalar manifold must induce linear transformations $M_i{}^j$ of the 1-forms' kinetic matrix $a_{ij}(\phi)$ that can be compensated by the inverse linear transformation acting on the 1-forms.

Defining the $\mathrm{SO}(n_2, n_2)$ generators by

$$M_M{}^N \sim \delta_M{}^N + \alpha^A T_{A\,M}{}^N\,, \tag{3.36}$$

we find that the above constraint implies

$$T_{A\,(MN)} \equiv T_{A\,(M}{}^P \eta_{N)P} = 0\,. \tag{3.37}$$

As discussed above, the same transformations must also act linearly on the 1-forms, and, therefore, we can define the generators in the corresponding representation:

$$M_i{}^j \sim \delta_i{}^j + \alpha^A T_{A\,i}{}^j\,. \tag{3.38}$$

In both representations, the generators T_A satisfy the same Lie algebra

$$[T_A, T_B] = -f_{AB}{}^C T_C\,. \tag{3.39}$$

Since (part of) the symmetry group can act trivially on either vectors or 2-forms we allow some of the generators T_A to be zero. It is for example possible that some symmetry generators act trivially on the 2-forms while they transform some of the scalars and vectors. In this case we have vanishing generators $T_{AM}{}^N$ and non-vanishing $T_{Ai}{}^j$. Still both (formally) satisfy the above algebra.

The ζ-tensor can be redefined in an $\mathrm{SO}(n_2, n_2)$-covariant way:

$$(\zeta^M{}_N) \equiv \begin{pmatrix} 0 & \zeta^{\Lambda\Sigma} \\ \zeta_{\Lambda\Sigma} & 0 \end{pmatrix}\,, \qquad \zeta_{\Lambda\Sigma} = \zeta^{\Lambda\Sigma}\,, \tag{3.40}$$

so the (anti-) self-duality constraint takes the form

$$\zeta^M{}_N (J^N - \zeta^N{}_P J^P) = 0\,. \tag{3.41}$$

3.2 Gaugings and massive deformations

In general the above theory will have a group of global symmetries G with constant parameters α^A. As discussed in the previous section, infinitesimally, these global symmetries act on the scalars ϕ^x, 1-forms A^i and electric and magnetic 2-forms B^M as

$$\delta_\alpha \phi^x = \alpha^A k_A{}^x(\phi)\,, \tag{3.42}$$

$$\delta_\alpha A^i = \alpha^A T_{A\,j}{}^i A^j\,, \tag{3.43}$$

$$\delta_\alpha B^M = \alpha^A T_{A\,N}{}^M B^N\,, \tag{3.44}$$

where the matrices $T_{A\,M}{}^{N}$ are generators of $\mathrm{SO}(n_2, n_2)$, i.e. they satisfy eq. (3.37), and the $k_A{}^x(\phi)$ are Killing vectors of the metric $g_{xy}(\phi)$. Some of the matrices and Killing vectors may be identically zero. They satisfy the algebras eq. (3.39) and $[k_A, k_B] = -f_{AB}{}^{C} k_C$.

These transformations will be global symmetries of the theory constructed in the previous section if the following five conditions are met:

1. The vectors $k_A{}^x(\phi)$ are Killing vectors of the metric $g_{xy}(\phi)$ of the scalar manifold.

2. The kinetic matrices $a_{ij}, f_{\Lambda\Sigma} \equiv b_{\Lambda\Sigma} + c_{\Lambda\Sigma}$ satisfy the conditions

$$\pounds_A a_{ij} = -2T_{A\,(i}{}^{k} a_{j)k}\,, \tag{3.45}$$

$$\pounds_A f_{\Lambda\Sigma} = -T_{A\,\Lambda\Sigma} + 2T_{A\,(\Lambda}{}^{\Omega} f_{\Sigma)\Omega} - T_A{}^{\Omega\Gamma} f_{\Omega\Lambda} f_{\Gamma\Sigma}\,, \tag{3.46}$$

   where $\pounds_A$ denotes the Lie derivative along the vector k_A and the matrices T_A are different components of some of the generators of $\mathrm{SO}(n_2, n_2)$ in the fundamental representation

$$M_N{}^M \sim \mathbb{I}_{2n_2\times 2n_2} + \alpha^A T_{A\,N}{}^{M} = \mathbb{I}_{2n_2\times 2n_2} + \alpha^A \begin{pmatrix} T_{A\,\Sigma}{}^{\Lambda} & T_A{}^{\Sigma\Lambda} \\ & \\ T_{A\,\Sigma\Lambda} & T_A{}^{\Sigma}{}_{\Lambda} \end{pmatrix}. \tag{3.47}$$

3. The deformation tensor $d_{M\,ij}$ is invariant

$$\delta_A d_{M\,ij} \equiv Y_{A\,Mij} = -T_{A\,M}{}^{N} d_{N\,ij} - 2T_{A\,(i}{}^{k} d_{M\,j)k} = 0\,. \tag{3.48}$$

4. The scalar potential is invariant

$$\pounds_A V = k_A V = 0\,. \tag{3.49}$$

5. The ζ-tensors is invariant

$$\delta_A \zeta^M{}_N = T_{A\,P}{}^{M} \zeta^P{}_N - T_{A\,N}{}^{P} \zeta^M{}_P = 0\,. \tag{3.50}$$

As we did in the 5-dimensional case, we will relax some of these conditions to construct a gauged theory. In the next section when we construct the tensor hierarchy and the action we only require invariance of $d_{M\,ij}$ under that subgroup of G that is gauged. Taking the limit in which all deformation tensors but $d_{M\,ij}$ vanish we recover the results of this section and in particular the action will generically only be invariant under a subgroup of G. The ζ-tensor on the other hand is not a deformation tensor and we therefore have the condition that it must be an invariant tensor of the symmetry group.

To gauge the theory we introduce, as in the 5-dimensional case, the embedding tensor $\vartheta_i{}^A$, subject to the quadratic constraint (eq. (2.20) with the indices I, J, K replaced by i, j, k) which reflects its gauge-invariance. Following the same steps as in the 5-dimensional case, we introduce the gauge-covariant derivative of the scalars eq. (2.19) and, from the Bianchi identity associated to it, eq. (2.24), we arrive at the definition of the 2-form field strength F^i given in eq. (2.28) up to the undetermined term ΔF^i subject to the condition eq. (2.29). Gauge-covariance of F^i implies the gauge transformation eq. (2.30) for ΔF^i, which we rewrite here for convenience:

$$\delta_\Lambda \Delta F^i = -\mathfrak{D}\Delta A^i + 2X_{(jk)}{}^i \left[\Lambda^j F^k + \frac{1}{2}A^j \wedge \delta_\Lambda A^k\right] . \tag{3.51}$$

In this case, in order to satisfy the constraint $\vartheta_i{}^A \Delta F^i = \vartheta_i{}^A \Delta A^i = 0$ it is natural to introduce a matrix Z^{iM} satisfying

$$Q^{AM} \equiv \vartheta_i{}^A Z^{iM} = 0 \,, \tag{3.52}$$

and define

$$\Delta F^i \equiv Z^{iM} B_M \,, \qquad \Delta A^i \equiv -Z^{iM}\Lambda_M \,, \tag{3.53}$$

where Λ_M is the 1-form gauge parameter under which the 2-forms B_M must transform. Then, the gauge transformation of ΔF^i implies

$$Z^{iM}\delta_\Lambda B_M = Z^{iM}\mathfrak{D}\Lambda_M + 2X_{(jk)}{}^i \left[\Lambda^j F^k + \frac{1}{2}A^j \wedge \delta_\Lambda A^k\right] . \tag{3.54}$$

This solution will only work if $X_{(jk)}{}^i \sim Z^{iM}\mathcal{O}_{M\,jk}$ for some tensor $\mathcal{O}_{M\,jk}$ symmetric in jk. It is natural to identify this tensor with the tensor $d_{M\,jk}$ that we know can be introduced in the physical theory so that

$$\delta_\Lambda B_M = \mathfrak{D}\Lambda_M + 2d_{M\,jk}\left[\Lambda^j F^k + \frac{1}{2}A^j \wedge \delta_\Lambda A^k\right] + \Delta B_M \,, \tag{3.55}$$

in which $Z^{iM}\Delta B_M = 0$. With this choice for we find agreement with what was found in the previous subsection obtained by setting $\vartheta_i{}^A = Z^{iM} = 0$.

We impose the constraint

$$Q_{jk}{}^i \equiv X_{(jk)}{}^i - Z^{iM} d_{M\,jk} = 0 \,, \tag{3.56}$$

where we have chosen the normalization of $d_{M\,jk}$ to recover the expression we got in the previous section. We thus find

$$F^i = dA^i + \frac{1}{2}X_{jk}{}^i A^{jk} + Z^{iM}B_M \,, \tag{3.57}$$

$$\delta_\Lambda A^i = -\mathfrak{D}\Lambda^i - Z^{iM}\Lambda_M \,, \tag{3.58}$$

$$\delta_\Lambda B_M = \mathfrak{D}\Lambda_M + 2d_{M\,kl}\left(\Lambda^k F^l + \frac{1}{2}A^k \wedge \delta_\Lambda A^l\right) + \Delta B_M \,, \qquad Z^{iM}\Delta B_M = 0 \,, \tag{3.59}$$

where the possible additional term ΔB_M will be determined by the requirement of gauge-covariance of the 3-form field strength H_M.

We must require the tensors Z^{iM} and $d_{M\,ij}$ to be gauge-invariant, which leads to the constraints

$$Q_i{}^{jM} \equiv -\delta_i Z^{jM} = -X_{ik}{}^j Z^{kM} - X_{iN}{}^M Z^{jN} = 0 \,, \tag{3.60}$$

$$Q_{i\,M\,jk} \equiv -\delta_i d_{M\,jk} = X_{i\,M}{}^N d_{N\,jk} + 2X_{i\,(j|}{}^l d_{M\,|k)l} = 0 \,. \tag{3.61}$$

This last constraint is clearly weaker than the global invariance constraint $Y_{A\,Mij} = 0$ in eq. (3.48).

3.2.1 The 3-form field strengths H_M

The covariant derivative of the 2-form field strengths F^i, after use of the generalized Jacobi identities[13] is

$$\mathfrak{D}F^i = Z^{iM}\left\{\mathfrak{D}B_M + d_{M\,jk}\left[A^j \wedge dA^k + \frac{1}{3}X_{lm}{}^k A^{jlm}\right]\right\}\,, \tag{3.62}$$

which leads us to define the 3-form field strength

$$\mathfrak{D}F^i = Z^{iM}H_M\,, \tag{3.63}$$

$$H_M \equiv \mathfrak{D}B_M + d_{M\,jk}\left[A^j \wedge dA^k + \frac{1}{3}X_{lm}{}^k A^{jlm}\right] + \Delta H_M\,, \tag{3.64}$$

$$Z^{iM}\Delta H_M = 0\,, \tag{3.65}$$

where ΔH_M will be determined, together with ΔB_M by using gauge-covariance of H_M, which is guaranteed by the formalism. To proceed with constructing the hierarchy we do not need the explicit form of the gauge transformations ΔB_M. Just as in the 5-dimensional case we can continue with constructing gauge-covariant field strengths by computing the Bianchi identities. The form of ΔH_M will be a contraction of some invariant tensor(s), that are annihilated by Z^{iM}, with some 3-forms. We will determine ΔH_M simultaneously with the 4-form field strengths G_i.

3.2.2 The 4-form field strengths G_i

The Bianchi identity of H_M takes the form

$$\begin{aligned}\mathfrak{D}H_M &= d_{M\,ij}F^{ij} + \mathfrak{D}\Delta H_M\\ &\quad + Z_{M\,i}{}^N\left\{\left(F^i - \frac{1}{2}Z^{iP}B_P\right)\wedge B_N + \frac{1}{3}d_{N\,jk}A^{ij}\wedge dA^k + \frac{1}{12}X_{jk}{}^n d_{N\,ln}A^{ijkl}\right\}\,,\end{aligned} \tag{3.66}$$

where we have defined the tensor

$$Z_{M\,i}{}^N \equiv -X_{i\,M}{}^N - 2d_{M\,ij}Z^{jN}\,, \tag{3.67}$$

which is annihilated by Z^{jM}, i.e. $Z^{jM}Z_{M\,i}{}^N = 0$ by virtue of eqs. (3.52), (3.56) and (3.60).

The simplest Ansatz we can make is to assume that $\Delta H_M = Z_{M\,i}{}^N C_N{}^i$ for some 3-forms $C_N{}^i$. However, in $d = 6$ dimensions the 3-forms of a physical theory are dual to the 1-forms, and, therefore, as we have shown in the case that $\vartheta_i{}^A = Z^{iM} = 0$, we can only have 3-forms C_i. This means that we must define a new[14] invariant tensor $Z_M{}^i$ such that

$$\Delta H_M = Z_M{}^i C_i\,, \qquad Z^{jM}Z_M{}^i = 0\,. \tag{3.68}$$

[13] In the 6-dimensional theory the generalized Jacobi identity reads $X_{[jk}{}^m X_{l]m}{}^i = \frac{2}{3}Z^i{}_N X_{[jk}{}^m d^N{}_{l]m}$.

[14] In principle $Z_M{}^i$ and $Z^i{}_M$ are unrelated, but we are going to see that we can relate these two tensors, though. This is not just an economical possibility, but reflects the fact that if a p-form has a stückelberg coupling to a $(p+1)$-form, then their duals, which will be, respectively, $(\tilde{p}+1)$- and $\tilde{p}$-forms (with $\tilde{p} = d-p-2$), will also have Stückelberg couplings with the same parameters and reversed roles: the $\tilde{p}$-form, dual of the $(p+1)$-form, will be the Stückelberg field of the $(\tilde{p}+1)$-form, dual of the p-form.

In order to make contact with the field strength G_i in eq. (3.23) of the theory obtained for $\vartheta_i{}^A = Z^{iM} = 0$ we must require

$$Z_{M\,i}{}^N = 2Z_M{}^j d^N{}_{ji}\,, \tag{3.69}$$

so that the Bianchi identity will take the form

$$\begin{aligned}
\mathfrak{D}H_M &= d_{M\,ij}F^i \wedge F^j + Z_M{}^i G_i\,,\\
G_i &= \mathfrak{D}C_i + 2d^N{}_{ip}\left[\left(F^p - \frac{1}{2}Z^{pM}B_M\right) \wedge B_N + \frac{1}{3}d_{N\,jk}A^{pj} \wedge dA^k + \frac{1}{12}X_{jk}{}^n d_{N\,ln}A^{pjkl}\right]\\
&\quad + \Delta G_i\,,\\
Z_M{}^i \Delta G_i &= 0\,.
\end{aligned} \tag{3.70}$$

The requirement (3.69) leads to

$$X_{i\,MN} = -2(d_{M\,ij}Z^j{}_N + d_{N\,ij}Z_M{}^j)\,. \tag{3.71}$$

The antisymmetry of $X_{i\,MN}$ suggests[15] to take

$$Z^{Mi} = -Z^{iM}\,. \tag{3.72}$$

Summarizing we have thus two new constraints:

$$Q_{i\,MN} \equiv X_{i\,MN} - 4Z^j{}_{[M}d_{N]ij} = 0\,, \tag{3.73}$$

$$Q^{ij} \equiv Z^{iM}Z^j{}_M = 0\,, \tag{3.74}$$

from which it follows that the tensor

$$C_{MNP} \equiv d_{M\,ij}Z^i{}_N Z^j{}_P\,, \tag{3.75}$$

is totally symmetric.

The constraint $Q^{ij} = 0$ is similar to the constraint $\vartheta_M{}^A \vartheta^{MB} = 0$ in 4 dimensions [11].

We will show the validity of this construction by proving the consistency of the resulting tensor hierarchy.

3.2.3 The 5-form field strengths K_A

If we take the covariant derivative of the Bianchi identity of H_M we find

$$Z_M{}^i[\mathfrak{D}G_i - 2d^N{}_{ij}F^j \wedge H_N] = 0\,, \tag{3.76}$$

from which it follows that the Bianchi identity of G_i must have the form

$$\mathfrak{D}G_i = 2d^N{}_{ij}F^j \wedge H_N + \Delta\mathfrak{D}G_i\,, \qquad Z_M{}^i \Delta\mathfrak{D}G_i = 0\,. \tag{3.77}$$

[15] See footnote 14.

A direct calculation using the above expression for G_i gives the result

$$\begin{aligned}
\mathfrak{D}G_i \,=\, & 2d^M{}_{ij}F^j \wedge H_M + \mathfrak{D}\Delta G_i + \vartheta_i{}^A \Big\{ T_A{}^{MN}\left(H_M - \frac{1}{2}\mathfrak{D}B_M\right) \wedge B_N \\
& +T_{A\,k}{}^p \Big[(F^k - Z^{kM}B_M) \wedge C_p - \frac{1}{6} d^M{}_{jp} d_{M\,lm} A^{jkl} \wedge dA^m \\
& +\frac{1}{30} X_{lm}{}^q d^M{}_{jq} d_{M\,pn} A^{jklmn}\Big]\Big\}\,,
\end{aligned} \tag{3.78}$$

up to terms proportional to the constraint eq. (3.26) which, so far we had not needed. The reason why we need to use it here is that the term $d_{M\,i(j}d^M{}_{kl)}$ is not annihilated by Z^{iN} and we cannot argue that it is proportional to $\vartheta_i{}^A$ times some new tensor. the only consistent way forward is to use eq. (3.26).

Since $Z^{iM}\vartheta_i{}^A = 0$, we can set $\Delta G_i = \vartheta_i{}^A D_A$ for some 4-forms D_A and write the Bianchi identity for the 4-form field strength G_i in the form

$$\mathfrak{D}G_i \,=\, 2d^M{}_{ij}F^j \wedge H_M + \vartheta_i{}^A K_A\,, \tag{3.79}$$

$$\begin{aligned}
K_A \,=\, & \mathfrak{D}D_A + T_A{}^{MN}\left(H_M - \frac{1}{2}\mathfrak{D}B_M\right) \wedge B_N \\
& +T_{A\,k}{}^p \Big[(F^k - Z^{kM}B_M) \wedge C_p - \frac{1}{6} d^M{}_{jp} d_{M\,lm} A^{jkl} \wedge dA^m + \frac{1}{30} X_{lm}{}^q d^M{}_{jq} d_{M\,pn} A^{jklmn}\Big] \\
& +\Delta K_A\,,
\end{aligned} \tag{3.80}$$

$$\vartheta_i{}^A \Delta K_A \,=\, 0\,. \tag{3.81}$$

3.2.4 The 6-form field strengths L

The covariant derivative of the Bianchi identity of G_i implies that the Bianchi identity for the 5-form field strengths must be of the form

$$\mathfrak{D}K_A = T_{A\,j}{}^k F^j \wedge G_k - \frac{1}{2} T_A{}^{MN} H_M \wedge H_N + \Delta\mathfrak{D}K_A\,, \qquad \vartheta_i{}^A \Delta\mathfrak{D}K_A = 0\,. \tag{3.82}$$

It is useful to have some idea of what we can expect concerning $\mathfrak{D}K_A$ according to the general formalism that we have introduced before.

As we have seen, 6-dimensional gauge theories are determined by three different deformation tensors $\vartheta_i{}^A, Z^{iM}, d_{M\,ij}$ satisfying the 5 constraints $Q = 0$:

$$Q^{AM} \equiv \vartheta_i{}^A Z^{iM}\,, \tag{3.83}$$

$$Q^{ij} \equiv Z^{iM} Z^j{}_M\,, \tag{3.84}$$

$$Q_{jk}{}^i \equiv X_{(jk)}{}^i - Z^{iM} d_{M\,jk}\,, \tag{3.85}$$

$$Q_{i\,MN} \equiv X_{i\,MN} - 4Z^j{}_{[M} d_{N]ij}\,, \tag{3.86}$$

$$Q_{ijk\,l} \equiv d_{M\,(ij} d^M{}_{k)l}\,, \tag{3.87}$$

plus the three constraints associated to the gauge-invariance of the deformation tensors:

$$Q_{ji}{}^A \equiv \; -\delta_j \vartheta_i{}^A \;= -\vartheta_j{}^B Y_{B\,i}{}^A = -\vartheta_j{}^B (f_{BC}{}^A \vartheta_i{}^C - T_{B\,i}{}^k \vartheta_k{}^A)\,, \tag{3.88}$$

$$Q_j{}^{iM} \equiv \; -\delta_j Z^{iM} \;= -\vartheta_j{}^A Y_A{}^{iM} = -\vartheta_j{}^A (T_{A\,k}{}^i Z^{kM} + T_{A\,N}{}^M Z^{iN})\,, \tag{3.89}$$

$$Q_{k\,M\,ij} \equiv -\delta_k d_{M\,ij} = -\vartheta_k{}^A Y_{A\,M\,ij} = \vartheta_k{}^A (2T_{A\,(i|}{}^l d_{M\,|j)l} + T_{A\,M}{}^N d_{N\,ij})\,. \tag{3.90}$$

We thus expect three 5-forms $E^i{}_A, E_{iM}, E^{M\,ij}$ dual to the deformation tensors that will appear in the field strength K_A through the term

$$\Delta K_A = Y_{A\,i}{}^B E^i{}_B + Y_A{}^{iM} E_{iM} + Y_{A\,M\,ij} E^{M\,ij} \,. \tag{3.91}$$

The result of a direct calculation is

$$\begin{aligned}
\mathfrak{D}K_A = {} & T_{A\,j}{}^k F^j \wedge G_k - \frac{1}{2} T_A{}^{MN} H_{MN} \\
& + Y_{A\,i}{}^B \Big\{ -F^i \wedge D_B + \frac{1}{30} T_{B\,k}{}^n d^N{}_{jm} d_{N\,ln} A^{ijkl} \wedge dA^m \\
& \qquad + \frac{1}{80} T_{B\,k}{}^p X_{lm}{}^q d^N{}_{jq} d_{N\,pn} A^{ijklmn} \Big\} \\
& + Y_A{}^{iM} \Big\{ (H_M - \mathfrak{D}B_M) \wedge C_i - B_M \wedge (G_i - \vartheta_i{}^B D_B) - \frac{1}{2} Z^j{}_M C_{ij} \\
& \qquad + d^N{}_{ij} F^j \wedge B_{MN} + \frac{1}{3} d^N{}_{ij} Z^{jP} B_{MNP} \Big\} \\
& + Y_A{}^M{}_{ij} \Big\{ -F^{ij} \wedge B_M + Z^{iN} F^j \wedge B_{MN} - \frac{1}{3} Z^{iN} Z^{jP} B_{MNP} \\
& \qquad - \frac{1}{2} d_{M\,kl} A^{ik} \wedge dA^{jl} - \frac{2}{15} X_{kl}{}^n d_{M\,nm} A^{iklm} \wedge dA^j - \frac{1}{5} X_{kl}{}^j d_{M\,nm} A^{ikln} \wedge dA^m \\
& \qquad - \frac{1}{18} X_{kl}{}^j X_{np}{}^q d^M{}_{mq} A^{iklmnp} \Big\} \\
& + \mathfrak{D}\Delta K_A \,.
\end{aligned} \tag{3.92}$$

If we take $\Delta\mathfrak{D}K_A$ to be

$$\Delta\mathfrak{D}K_A = Y_{A\,i}{}^B L^i{}_B + Y_A{}^{iM} L_{iM} + Y_{A\,M\,ij} L^{M\,ij} \,, \tag{3.93}$$

where $L^i{}_B$, L_{iM}, $L^{M\,ij}$ are the gauge-covariant field strengths of the 5-forms $E^i{}_B$, E_{iM} and $E^{M\,ij}$, respectively, then we obtain the Bianchi identity for K_A given in eq. (C.19) with the 6-form field strengths $L^i{}_B$, L_{iM}, $L^{M\,ij}$ given in eqs. (C.13), (C.14) and (C.15).

In eqs. (C.13), (C.14) and (C.15) we have not specified in detail the Stückelberg couplings to the 6-forms that we denoted by $F_\flat$. There are in total eight top-forms in 6-dimensions corresponding to the eight constraints. These eight top-forms are determined up to massive gauge transformations of the form $\delta F_\flat = \Sigma_\flat$ such that $W_\sharp{}^\flat \Sigma_\flat = 0$. This is because all the top-forms only come contracted with $W_\sharp{}^\flat$. In particular theories it can happen that these massive gauge transformations enable one to complete gauge away certain top-forms entirely. The massless gauge transformations of the top-forms contain the 5-form gauge transformation parameter $\Lambda_\flat$, i.e. $W_\sharp{}^\flat \delta F_\flat = W_\sharp{}^\flat \mathfrak{D}\Lambda_\flat$. This parameter also shows up in the gauge transformation of the 5-form potentials $E_\sharp$ as $\delta E_\sharp = -W_\sharp{}^\flat \Lambda_\flat$. Depending on the details of the theory these massive gauge transformation may allow one to entirely gauge away certain 5-forms.

3.2.5 Gauge-invariant action for the 1-, 2- and 3-forms

Our starting point to construct a 6-dimensional gauge-invariant action is[16]

$$S_1 \equiv \int \Big\{ \frac{1}{2} g_{xy}(\phi) \mathfrak{D}\phi^x \wedge \star \mathfrak{D}\phi^y - \frac{1}{2} a_{ij}(\phi) F^i \wedge \star F^j \\ + \frac{1}{2} b_{\Lambda\Sigma}(\phi) H^\Lambda \wedge \star H^\Sigma + \frac{1}{2} c_{\Lambda\Sigma}(\phi) H^\Lambda \wedge H^\Sigma + \star V(\phi) \Big\} , \tag{3.94}$$

where the covariant derivative and field strengths are those of the tensor hierarchy. This means, in particular, that

$$\mathfrak{D} B^\Sigma = dB^\Sigma + X_{i\,M}{}^\Sigma A^i \wedge B^M , \tag{3.95}$$

so the magnetic 2-forms B_Σ occur in this action.

As a general rule, the gauge-invariant action will only differ from this one by topological Chern-Simons-like terms. Furthermore, the equations of motion will just be gauge-covariant generalizations of the ungauged ones, up to duality transformations. More precisely, as a general rule, the equations of motion of the magnetic higher-rank form fields (here the magnetic 2-forms B_Σ and the 3-forms C_i) will just be duality relations, and the equations of motion of the (electric) lower-rank potentials (here the 1-forms A^i and the electric 2-forms B^Σ) will be completely equivalent to the hierarchy's Bianchi identities after use of the duality relations.

Let us first consider all those which contain the 3-forms C_i. Taking into account that we expect the equation of motion of C_i to be a duality relation for the 3-form field strengths, a reasonable Ansatz for the terms that involve 3-forms is

$$S_2 \equiv \int Z^{i\Sigma} C_i \wedge \left(H_\Sigma + \frac{1}{2} Z^j{}_\Sigma C_j \right) , \tag{3.96}$$

since, if we only vary w.r.t. the 3-forms, we get

$$\delta(S_1 + S_2) = -Z^{iM} \delta C_i \wedge [J_M - H_M] , \tag{3.97}$$

where J_Λ is given in eq. (3.11) (but with the field strengths H^Λ replaced by those of the hierarchy) and the upper component of the doublet J^M is defined to be $J^\Sigma \equiv H^\Sigma$.

Let us now consider the topological terms containing magnetic 2-forms B_Λ. We expect the equations of motion of the B_Λ to give the duality relation between 2- and 4-form field strengths (up to, possibly, other duality relations). If we only vary B_Σ in $S_1 + S_2$ we find the result

$$\delta(S_1 + S_2) = \delta B_\Sigma \wedge \left\{ -Z^{i\Sigma} [a_{ij} \star F^j - \mathfrak{D} C_i] + X_i{}^{\Sigma\Omega} A^i \wedge [J_\Omega + Z^j{}_\Omega C_j] \right\} , \tag{3.98}$$

whose two terms have the form of incomplete duality relations, in agreement with our prejudice. If we require that the next term we add to the action, S_3, gives, upon variation of B_Σ only, the complete duality relations

$$\delta(S_1 + S_2 + S_3) = -\delta B_\Sigma \wedge \left\{ Z^{i\Sigma} [a_{ij} \star F^j - G_i] + \mathfrak{D}(J^\Sigma - H^\Sigma)] \right\} , \tag{3.99}$$

[16]We do not consider the Einstein-Hilbert term as it plays no role in the discussion.

we find that

$$S_3 \equiv \int \Big\{ B_\Sigma \wedge \Big\{ Z^{i\Sigma} \Big[2d_{\Omega\, ij} f^j \wedge B^\Omega + g_i + \frac{1}{2} d^\Omega{}_{ij} X_{kl}{}^j A^{kl} \wedge B_\Omega \Big]$$
$$+2d^\Sigma{}_{ij} Z^{j\Omega} A^i \wedge dB_\Omega + X_i{}^{\Sigma\Omega} A^i \wedge \Big[-h_\Omega + X_{j\,\Omega\Gamma} A^j \wedge B^\Gamma + \frac{1}{2} X_{j\,\Omega}{}^\Gamma A^j \wedge B_\Gamma \Big] \Big\}$$
$$+\frac{1}{3} d^M{}_{ij} Z^{iN} Z^{jP} B_{MNP} - \frac{1}{3} d_{\Lambda\, ij} Z^i{}_\Sigma Z^j{}_\Omega B^{\Lambda\Sigma\Omega} \Big\}, \tag{3.100}$$

where f^j, h_Ω and g_i are, respectively, the part of the field strengths F^j, H_Ω and G_i that only depend on the 1-forms A^i, i.e.

$$f^j \equiv dA^j + \frac{1}{2} X_{kl}{}^j A^{kl}, \tag{3.101}$$

$$h_M \equiv d_{M\,jm} A^j \wedge dA^m + \frac{1}{3} d_{M\,jm} X_{kl}{}^m A^{jkl}, \tag{3.102}$$

$$g_i \equiv \frac{2}{3} d^M{}_{ij} d_{M\,kl} A^{jk} \wedge dA^l + \frac{1}{6} d^M{}_{ij} d_{M\,kl} X_{mn}{}^l A^{jkmn}. \tag{3.103}$$

Observe that S_3 does not contain any 3-forms and, therefore, the variation of the action w.r.t. the 3-forms, eq. (3.97), does not change when we add S_3.

We next consider the variations w.r.t. the electric 2-forms B^Σ. These should give the equations of motion of the electric 2-forms up to duality relations. Adding

$$S_4 \equiv \int \Big\{ d_{\Sigma\, ij} B^\Sigma \wedge f^{ij} + \frac{1}{3} d_{\Lambda\, ij} Z^i{}_\Sigma Z^j{}_\Omega B^{\Lambda\Sigma\Omega}$$
$$+X_{i\,\Sigma\Omega} A^i \wedge h^\Omega \wedge B^\Sigma + 2d_{\Sigma\, ij} Z^i{}_\Omega A^j \wedge dB^\Sigma \wedge B^\Omega$$
$$+\frac{1}{2} \left(d_{\Sigma\, ij} Z^i{}_\Omega X_{kl}{}^j - X_{k\,\Sigma\Gamma} X_l{}^\Gamma{}_\Omega \right) A^{kl} \wedge B^{\Sigma\Omega} \Big\}, \tag{3.104}$$

we find that varying only w.r.t. B^Σ gives

$$\delta(S_1 + S_2 + S_3 + S_4) = -\delta B^\Sigma \wedge \{ Z^i{}_\Sigma [a_{ij} \star F^j - G_i] + \mathfrak{D}(J_\Sigma - H_\Sigma) \}, \tag{3.105}$$

which, upon duality relations gives the hierarchy's Bianchi identity of the magnetic 3-form field strengths H_Σ. S_4 does not contain any 3-forms or magnetic 2-forms and, therefore, adding S_4 does not change neither eq. (3.97) nor eq. (3.99).

Finally, let us consider the variation of S_1 w.r.t. the 1-forms A^i only. We can write the result in the form

$$\delta S_1 = \delta A^i \wedge \Big\{ -\star \frac{\delta S}{\delta A^i} + s_i \Big\}, \tag{3.106}$$

where we have *defined*

$$\star \frac{\delta S}{\delta A^i} \equiv \mathfrak{D}(a_{ij} \star F^j) - 2d^M{}_{ij} F^j \wedge J_M - \vartheta_i{}^A \star j_A$$
$$+d^M{}_{ij} A^j \wedge \Big[Z^k{}_M (a_{kl} \star F^l - G_k) + \mathfrak{D}(J_M - H_M) \Big]$$
$$+ \Big[2d_{N\,il} B^N + \frac{2}{3} d^N{}_{lj} d_{N\,ki} A^{jk} \Big] \wedge Z^{lM} [J_M - H_M], \tag{3.107}$$

and

$$
\begin{aligned}
s_i \equiv & -d_{\Sigma\, ij} Z^{k\Sigma} A^j \wedge G_k - (2d_{\Sigma\, ij} F^j + d^{\Omega}{}_{ij} X_{k\,\Omega\Sigma} A^{jk}) \wedge H^{\Sigma} \\
& + [X_{i\,M}{}^{\Sigma} B^M + d^{\Sigma}{}_{l[i} X_{jk]}{}^{l} A^{jk} - d^{\Omega}{}_{ij} X_{k\,\Omega}{}^{\Sigma} A^{jk} - 2d^{\Sigma}{}_{ij}(F^j - dA^j)] \wedge H_{\Sigma} \\
& - d^{\Sigma}{}_{ij} d_{\Sigma\, kl} A^j \wedge F^{kl} \,.
\end{aligned} \tag{3.108}
$$

While this definition is mainly based on intuition, we can check that the variations of the pieces S_2, S_3 and S_4 w.r.t. A^i only contribute to s_i: the variation of S_2 w.r.t. A^i cancels all the terms in s_i containing the 3-forms C_i; the variation of S_3 w.r.t. A^i cancels all the terms in s_i containing the magnetic 2-forms B_Σ and the variation of S_4 w.r.t. A^i cancels all the terms in s_i containing the electric 2-forms B^Σ, leaving unchanged what we have defined as $\frac{\delta S}{\delta A^i}$. Thus, we only need to see if there exists an S_5 whose variation w.r.t. A^i cancels the terms in s_i that only depend on the 1-forms A^i. In other words: we have to determine the integrability of the terms in $\delta A^i \wedge s_i$ that only depend on 1-forms. This highly non-trivial requirement is satisfied and S_5 is given by

$$
\begin{aligned}
S_5 = & \frac{1}{4} \left[d_{\Sigma\, ik} d^{\Sigma}{}_{jl} - d^{\Sigma}{}_{ik} d_{\Sigma\, jl} \right] A^{ij} \wedge dA^{kl} \\
& + X_{ij}{}^{p} \left[\frac{2}{15} d_{\Sigma\, km} d^{\Sigma}{}_{lp} - \frac{1}{5} d^{\Sigma}{}_{km} d_{\Sigma\, lp} \right] A^{ijkl} \wedge dA^m \\
& + \frac{1}{9} \left[d_{\Sigma\, ip} d^{\Sigma}{}_{jq} + \frac{1}{2} d^{\Sigma}{}_{ip} d_{\Sigma\, jq} \right] X_{kl}{}^{p} X_{mn}{}^{q} A^{ijklmn} \,.
\end{aligned} \tag{3.109}
$$

It is evident that this additional term does not modify the variations of the total action[17]

$$
S \equiv S_1 + \cdots + S_5 \tag{3.110}
$$

w.r.t. the 3- and 2-forms.

We, thus arrive at the following result:

$$
\begin{aligned}
\delta S = \int \Big\{ & -\delta\phi^x \star \frac{\delta S}{\delta \phi^x} - \delta A^i \wedge \star \frac{\widetilde{\delta S}}{\delta A^i} - (\delta B^M - d^M{}_{ij} A^i \wedge \delta A^j) \wedge \star \frac{\delta S}{\delta B^M} \\
& - \left[\delta C_i + 2 d_{M\, ij} B^M \wedge \delta A^j + \frac{2}{3} d^M{}_{ij} d_{M\, kl} A^{jk} \wedge \delta A^l \right] \wedge \frac{\delta S}{\delta C_i} \Big\} \,,
\end{aligned} \tag{3.111}
$$

where

$$
\begin{aligned}
\star \frac{\delta S}{\delta \phi^x} &= g_{xy} \mathfrak{D} \star \mathfrak{D} \phi^y + \frac{1}{2} \partial_x a_{ij} F^i \wedge \star F^j - \frac{1}{2} H^M \wedge \partial_x J_M - \star \partial_x V \,, && (3.112) \\
\frac{\delta S}{\delta C_i} &= Z^{iM} (J_M - H_M) \,, && (3.113) \\
\star \frac{\delta S}{\delta B^M} &= Z^i{}_M (a_{ij} \star F^j - G_i) + \mathfrak{D}(J_M - H_M) \,, && (3.114) \\
\star \frac{\widetilde{\delta S}}{\delta A^i} &= \mathfrak{D}(a_{ij} \star F^j) - 2 d^M{}_{ij} F^j \wedge J_M - \vartheta_i{}^A \star j_A \,. && (3.115)
\end{aligned}
$$

[17] A similar action for the case of the maximal 6-dimensional supergravity theory was constructed in [16].

We can now relate the equations of motion derived from this action and the tensor hierarchy's Bianchi identities via the duality relations

$$a_{ij}\star F^j = G_i\,, \tag{3.116}$$
$$J_M = H_M\,, \tag{3.117}$$
$$\star j_A = K_A\,, \tag{3.118}$$
$$\star\frac{\partial V}{\partial c^\sharp} = L_\sharp\,. \tag{3.119}$$

With these duality relations, the 3-form and magnetic 2-form equations of motion are automatically solved. The electric 2-form equations of motion become the hierarchy Bianchi identity of the magnetic 2-forms. The 1-form equations of motion become the hierarchy's Bianchi identity of the 4-form field strengths G_i. The projected scalar equations of motion $k_A{}^x\star\frac{\delta S}{\delta\phi^x}$ become the hierarchy's Bianchi identity of the 5-form field strengths K_A if we use that $k_A a_{ij} = -2T_{A\,(i}{}^k a_{j)k}$ as well as $H^M\wedge k_A J_M = -T_{A\,M}{}^N J^M\wedge J_N$, the Killing property of the $k_A{}^x$ and the fact that

$$k_A V = \sum_\sharp Y_A{}^\sharp\frac{\partial V}{\partial c^\sharp}\,. \tag{3.120}$$

In section 3.1.1 we discussed the possibility of having (anti-)self dual 2-forms and we found that this can be described by the tensor $\zeta^M{}_N$. We could ask the same question now in the context of a gauged theory with massive deformations. The (anti-)self duality can again be written as

$$\zeta^M{}_N(J^N - \zeta^N{}_P J^P) = 0\,, \tag{3.121}$$

where now J^N contains the hierarchy field strengths H^M. This condition must be consistent with the equations of motion. After hitting the condition with a covariant derivative we find the following consistency conditions: eq. (3.17) and

$$\zeta^M{}_N(Z^{iN} - \zeta^N{}_P Z^{iP}) = 0\,. \tag{3.122}$$

The ζ-tensor is not predicted by the tensor hierarchy because it cannot distinguish between (A)SD or non-(A)SD 2-forms. This concept only exists once equations of motion are defined.

The gauge transformations that leave the action invariant can be written as

$$\delta A^i = -\mathfrak{D}\Lambda^i - Z^{iM}\Lambda_M\,, \tag{3.123}$$
$$\delta B_M = \mathfrak{D}\Lambda_M + 2d_{M\,ij}\left(\Lambda^i F^j + \frac{1}{2}A^i\wedge\delta A^j\right) - Z_M{}^i\Lambda_i + \Delta B_M\,, \tag{3.124}$$
$$\begin{aligned}\delta C_i &= \mathfrak{D}\Lambda_i + 2d_{N\,ij}\Lambda^j J^N - 2d_{N\,ij}\Lambda^N\wedge F^j\\ &\quad -2d_{N\,ij}B^N\wedge\delta A^j - \frac{2}{3}d^N{}_{ij}d_{N\,kl}A^{jk}\wedge\delta A^l\,.\end{aligned} \tag{3.125}$$

Potential	Gauge transformation	Interpretation (field strength dual to)	Stückelberg pair with	Existence
B_M	massive	J_M	$Z^i{}_M C_i$	$\forall M : Z^i{}_M \neq 0$
$Z^{iM} B_M$	massless	$Z^{iM} J_M$	ungauged A^i	$\forall i : Z^{iM} \neq 0$
B_M	massless	J_M	none	$\forall M : Z^{iM} = 0$
C_i	massive	$a_{ij} F^j$	$\vartheta_i{}^A D_A$	$\forall i : \vartheta_i{}^A \neq 0$
$Z^i{}_M C_i$	massless	$Z^i{}_M a_{ij} F^j$	B_M	$\forall M : Z^i{}_M \neq 0$
C_i	massless	$a_{ij} F^j$	none	$\forall i : \vartheta_i{}^A = Z^i{}_M = 0$
D_A	massive	current j_A of symmetry broken by V	$Y_A{}^\sharp E_\sharp$	$\forall A : Y_A{}^\sharp \neq 0$
$\vartheta_i{}^A D_A$	massless	current j_A of gauged symmetry	C_i	$\forall i : \vartheta_i{}^A \neq 0$
D_A	massless	current j_A of global symmetry	none	$\forall A : Y_A{}^\sharp = \vartheta_i{}^A = 0$
$E_\sharp$	massive	$\partial V / \partial c^\sharp$	$W_\sharp{}^\flat F_\flat$	$\forall \sharp : W_\sharp{}^\flat \neq 0$
$Y_A{}^\sharp E_\sharp$	massless	$Y_A{}^\sharp \partial V / \partial c^\sharp$	D_A	$\forall A : Y_A{}^\sharp \neq 0$
$E_\sharp$	massless	$\partial V / \partial c^\sharp$	none	$\forall \sharp : W_\sharp{}^\flat = Y_A{}^\sharp = 0$
$W_\sharp{}^\flat F_\flat$	massless	enforces constraints	$E_\sharp$	$\forall \sharp : W_\sharp{}^\flat \neq 0$

Table 2. All the $p \geq 2$ forms of the 6-dimensional tensor hierarchy, their Stückelberg properties and physical interpretation.

To prove this we only need the following Noether identities associated to the invariance under gauge transformations whose parameters are, respectively Λ^i, Λ^M and Λ_i,

$$\mathfrak{D} \star \frac{\widetilde{\delta S}}{\delta A^i} + \vartheta_i{}^A k_A{}^x \star \frac{\delta S}{\delta \phi^x} + 2 d^M{}_{ij} F^j \wedge \star \frac{\delta S}{\delta B^M} + 2 d_{M\,ij} J^M \wedge \frac{\delta S}{\delta C_j} = 0\,, \tag{3.126}$$

$$\mathfrak{D} \star \frac{\delta S}{\delta B^M} - Z^i{}_M \star \frac{\widetilde{\delta S}}{\delta A^i} - 2 d_{M\,ij} F^i \wedge \frac{\delta S}{\delta C_j} = 0\,, \tag{3.127}$$

$$\mathfrak{D} \frac{\delta S}{\delta C_i} - Z^{iM} \star \frac{\delta S}{\delta B^M} = 0\,. \tag{3.128}$$

We note that these gauge transformations are exactly those of the hierarchy except for the 3-form gauge transformation eq. (3.125) which can be written as

$$\delta C_i = \delta_h C_i + 2 d_{N\,ij} \Lambda^j (J^N - H^N)\,, \tag{3.129}$$

in which $\delta_h C_i$ (together with the 1-form δA^i and 2-form gauge transformations δB_M) is the gauge transformation under which H^M transforms gauge-covariantly.

We end this section by giving an overview in table 2 of the 6-dimensional tensor hierarchy and its physical interpretation. The way in which table 2 should be read is entirely analogous to the 5-dimensional case discussed at the end of section 2.2.5.

4 Discussion

Without making reference to any particular details of a 5- or 6-dimensional field theory we have constructed the tensor hierarchies for such theories and the corresponding gauge-invariant actions. We have found the dualities that relate these two structures.

Our results, together with those of refs. [11, 12] reveal a number of generic features that must be common to all tensor hierarchies:

1. The field content of a particular tensor hierarchy provides an exhaustive list of all possible potentials that one can introduce into a theory. The generic tensor hierarchies that we have constructed provide a minimal list. Depending on the existence of additional theory-specific constraints (as in the $N = 1, d = 4$ supergravity case), more potentials may be included.

2. In general, the deformation parameters of any field theory[18] are of three different kinds:

   (a) The embedding tensor ϑ, which determines the gauge group and gauge couplings.

   (b) The Stückelberg tensors Z that will determine the couplings between p-forms and $(p + 1)$-forms and between their respective duals, the $(\tilde{p} + 1)$- and $\tilde{p}$-forms (with $\tilde{p} = d - p - 2$).

   (c) The Chern-Simons tensors d which determine the Chern-Simons terms in the field strengths and action.

3. As explained in the introduction, the tensor hierarchy will contain one $(d - 1)$-form potential ("de-form") conjugate to each deformation parameter. In a democratic formulation, the de-forms will enforce the constancy of the corresponding deformation parameters. There may be additional top-forms associated to theory-specific constraints which cannot be studied in our generic models. It is unclear if there might be additional top-forms whose gauge transformations are unconnected to the hierarchy.[19]

4. These deformation parameters will be subject to four generic kinds of constraints:

   (a) Constraints that enforce the gauge-invariance of all deformation tensors: $\delta\vartheta = 0\,,\ \delta Z = 0\,,\ \delta d = 0$. The first of these is the standard *quadratic constraint* of the literature.

   (b) Orthogonality constraints between the embedding tensor and the first Stückelberg tensor $\vartheta \cdot Z = 0$ and between each Stückelberg tensor and the next one $Z \cdot Z' = 0$.

[18] In this list we are obviously leaving aside deformations such as the cosmological constant in non-supersymmetric theories, which are unrelated to massive or massless gauge symmetries. These deformation parameters do not couple to the hierarchy's p-form potentials and, therefore, are unaccounted for by it.

[19] What is also still an open question is how to construct the tensor hierarchy of a theory without vectors such as the type IIB supergravity theory.

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(c) Constraints that relate the X matrices with the Chern-Simons and Stückelberg or embedding tensors: $X \sim Z \cdot d = 0$. The so-called linear or representation constraint of the 4-dimensional theories can be viewed as an example of this kind of constraints.

(d) Constraints between products of Chern-Simons tensors $d \cdot d = 0$.

5. As explained in the introduction, the tensor hierarchy will contain a top-form potential conjugate to each of the constraints satisfied by the deformation tensors. In a democratic formulation, these top-form potentials will enforce the corresponding constraints.

6. In d-dimensions, a gauge-invariant action for the physical theory can be constructed using just the forms of rank 1 to $[d/2]$ (i.e. 2 in $d = 4, 5$ and 3 in $d = 6, 7$ etc.). The gauge transformations will be identical to those of the tensor hierarchy up to duality relations. These duality relations are essential to relate the tensor hierarchy to the physical theory and fix the way all the fields appear in the Lagrangian except for those scalars that are not participating in isometry currents.

A tensor hierarchy together with a set of duality relations for its field strengths (a structure called duality hierarchy in ref. [11]) is clearly a powerful tool to construct the most general bosonic field theory in a particular dimension. This can then be used as a starting point for the construction of more general supergravity theories by subsequently supersymmetrizing the hierarchy.

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A Conventions and some formulae

We use mostly-minus signature both in 5- and 6-dimensions.

p-forms are normalized as follows

$$\omega \equiv \frac{1}{p!}\omega_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \,. \tag{A.1}$$

The exterior product of a p-form ω and a q-form η is

$$\omega \wedge \eta \equiv \frac{1}{p!q!}\omega_{\mu_1\cdots\mu_p}\eta_{\nu_1\cdots\nu_q} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_q} \,, \tag{A.2}$$

so, its components are

$$(\omega \wedge \eta)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \cdots \mu_p} \eta_{\mu_{p+1} \cdots \mu_{p+q}]} \,. \tag{A.3}$$

The exterior derivative of a p-form ω is

$$d\omega \equiv \frac{1}{p!} \partial_\nu \omega_{\mu_1 \cdots \mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \,, \tag{A.4}$$

so, its components are

$$(d\omega)_{\mu_1 \cdots \mu_{p+1}} = (p+1) \partial_{[\mu_1} \omega_{\mu_2 \cdots \mu_{p+1}]} \,. \tag{A.5}$$

The d-dimensional volume form is, with mostly minus signature,

$$\sqrt{|g|} d^d x \equiv \frac{(-1)^{d-1}}{d! \sqrt{|g|}} \epsilon_{\mu_1 \cdots \mu_d} dx^{\mu_1} \wedge \cdots dx^{\mu_d} \,, \tag{A.6}$$

where we have defined the completely antisymmetric symbol such that (in curved indices)

$$\epsilon^{01 \cdots (d-1)} = +1 \,, \qquad \epsilon_{01 \cdots (d-1)} = g \equiv \det g = (-1)^{d-1} |g| \,. \tag{A.7}$$

The components of the Hodge dual of a p-form ω are defined by

$$(\star\omega)_{\mu_1 \cdots \mu_{d-p}} \equiv \frac{1}{p! \sqrt{|g|}} \epsilon_{\mu_1 \cdots \mu_{d-p} \nu_1 \cdots \nu_p} \omega^{\nu_1 \cdots \nu_p} \,, \tag{A.8}$$

so

$$\star\, \omega = \frac{1}{p!(d-p)! \sqrt{|g|}} \epsilon_{\mu_1 \cdots \mu_{d-p} \nu_1 \cdots \nu_p} \omega^{\nu_1 \cdots \nu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{d-p}} \,. \tag{A.9}$$

Then, for p-forms ω in d dimensions, with mostly minus signature,

$$\star^2\, \omega = (-1)^{d-1+p(d-p)}\, \omega \,. \tag{A.10}$$

It follows that for 3-forms H in 6 dimensions we have $\star^2 = +1$ so that we can have real self- and anti-self-dual 3-forms $H^\pm$

$$H^\pm \equiv \frac{1}{2}(H \pm \star H) \,, \qquad \star H^\pm = \pm H^\pm \,. \tag{A.11}$$

A d-form Ω in d-dimensions is always proportional to the volume form. We can always write

$$\begin{aligned} \Omega &= K \sqrt{|g|}\, d^d x \,, \\ K &= \frac{1}{d! \sqrt{|g|}} \epsilon^{\mu_1 \cdots \mu_d} \Omega_{\mu_1 \cdots \mu_d} \,. \end{aligned} \tag{A.12}$$

Using this property, we find the following formulae in d dimensions

$$\star\, R = (-1)^{d-1} R \sqrt{|g|} d^d x \,, \tag{A.13}$$

$$d\phi \wedge \star d\phi = (\partial\phi)^2 \sqrt{|g|}\, d^d x \,, \tag{A.14}$$

$$F \wedge \star F = \frac{(-1)^{d-1}}{2} F^2 \sqrt{|g|}\, d^d x \,, \tag{A.15}$$

$$H \wedge \star H = \frac{1}{3!} H^2 \sqrt{|g|}\, d^d x \,, \tag{A.16}$$

$$H \wedge \star \tilde{H} = \frac{1}{3!} H_{\mu\nu\rho} (\star\tilde{H})^{\mu\nu\rho} \sqrt{|g|}\, d^d x \,. \tag{A.17}$$

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B Summary of the general 5-dimensional tensor hierarchy

B.1 Deformation tensors and constraints

The deformation tensors of 5-dimensional field theories are $\vartheta_I{}^A$, $Z^{IJ} = Z^{[IJ]}$ and $C_{IJK} = C_{(IJK)}$. They are subject to the constraints

$$Q_{IJ}{}^A = -\vartheta_I{}^B Y_{BJ}{}^A = -\vartheta_I{}^B(\vartheta_J{}^C f_{BC}{}^A - T_{BJ}{}^K \vartheta_K{}^A)\,, \tag{B.1}$$

$$Q_I{}^{JK} = -\vartheta_I{}^A Y_A{}^{JK} = 2\vartheta_I{}^A T_{AL}{}^{[J} Z^{K]L}\,, \tag{B.2}$$

$$Q_{IJKL} = -\vartheta_I{}^A Y_{AJKL} = 3\vartheta_I{}^A T_{A(J}{}^M C_{KL)M}\,, \tag{B.3}$$

which express the gauge-invariance of the deformation tensors and

$$Q^{AI} = \vartheta_J{}^A Z^{JI}\,, \tag{B.4}$$

$$Q_{JK}{}^I = X_{(JK)}{}^I - Z^{IL} C_{JKL}\,. \tag{B.5}$$

B.2 Field strengths and Bianchi identities

The tensor hierarchies of general 5-dimensional bosonic field theories have 1-forms A^I, 2-forms B_I, 3-forms C_A, 4-forms $D^I{}_B$, D_{IJ}, D^{IJK} and 5-forms $E^{IJ}{}_A$, $E^I{}_{JK}$, E^{IJKL}, E_{AI} and $E^{IJ}{}_K$. The field strengths of the 1-, 2-, 3- and 4-form fields are given by

$$F^I = dA^I + \frac{1}{2} X_{JK}{}^I A^{JK} + Z^{IJ} B_J\,, \tag{B.6}$$

$$H_I = \mathfrak{D} B_I + C_{IJK} A^J \wedge dA^K + \frac{1}{3} C_{IM[J} X_{KL]}{}^M A^{JKL} + \vartheta_I{}^A C_A\,, \tag{B.7}$$

$$\begin{aligned} G_A = {}& \mathfrak{D} C_A + T_{AK}{}^I \left[\left(F^K - \frac{1}{2} Z^{KL} B_L\right) \wedge B_I + \frac{1}{3} C_{ILM} A^{KL} \wedge dA^M \right. \\ & \left. + \frac{1}{12} C_{ILP} X_{MN}{}^P A^{KLMN}\right] + Y_A{}^{IJ} D_{IJ} + Y_{AI}{}^B D_B{}^I + Y_{AIJK} D^{IJK}\,, \end{aligned} \tag{B.8}$$

$$\begin{aligned} K^I{}_B = {}& \mathfrak{D} D^I{}_B + (F^I - Z^{IL} B_L) \wedge C_B + \frac{1}{12} T_{BJ}{}^M C_{KML} A^{IJK} \wedge dA^L \\ & + \frac{1}{60} T_{BJ}{}^N C_{KPN} X_{LM}{}^P A^{IJKLM} + W_B{}^I{}_{KJ}{}^D E^{KJ}{}_D - Z^{IJ} E_{BJ} - T_{BK}{}^J E_J{}^{IK} \\ & - Y_B{}^{JK} E^I{}_{JK}\,, \end{aligned} \tag{B.9}$$

$$\begin{aligned} K_{IJ} = {}& \mathfrak{D} D_{IJ} - \left[H_{[I} - \frac{1}{2} \mathfrak{D} B_{[I}\right] \wedge B_{J]} + 2 X_{K[I}{}^L E^K{}_{J]L} - C_{KL[I} E^{KL}{}_{J]} \\ & - \vartheta_{[I|}{}^A E_{A|J]}\,, \end{aligned} \tag{B.10}$$

$$\begin{aligned} K^{IJK} = {}& \mathfrak{D} D^{IJK} + \frac{1}{3} A^{(I} \wedge dA^{JK)} + \frac{1}{4} X_{LM}{}^{(K} A^{I|LM} \wedge dA^{|J)} \\ & + \frac{1}{20} X_{LM}{}^{(J} X_{NP}{}^K A^{I)LMNP} + 3 X_{LM}{}^{(I|} E^{L|JK)M} + Z^{L(I} E_L{}^{JK)}\,, \end{aligned} \tag{B.11}$$

and are related by the Bianchi identities

$$\mathfrak{D} F^I = Z^{IJ} H_J\,, \tag{B.12}$$

$$\mathfrak{D} H_I = C_{IJK} F^{JK} + \vartheta_I{}^A G_A\,, \tag{B.13}$$

$$\mathfrak{D} G_A = T_{AK}{}^I F^K \wedge H_I + Y_A{}^{IJ} K_{IJ} + Y_{AI}{}^B K^I{}_B + Y_{AIJK} K^{IJK}\,. \tag{B.14}$$

B.3 Duality relations

$$H_I = a_{IJ} \star F^J \,, \tag{B.15}$$
$$G_A = \star j_A \,, \tag{B.16}$$
$$K_\sharp = \star \frac{\partial V}{\partial c^\sharp} \,. \tag{B.17}$$

C Summary of the general 6-dimensional tensor hierarchy

C.1 Deformation tensors and constraints

The deformation tensors of 6-dimensional field theories are $\vartheta_i{}^A$, Z^{iM} and $d_{M\,ij} = d_{M\,(ij)}$. They are subject to the constraints

$$Q_{ji}{}^A \equiv -\vartheta_j{}^B Y_{B\,i}{}^A = -\vartheta_j{}^B (f_{BC}{}^A \vartheta_i{}^C - T_{B\,i}{}^k \vartheta_k{}^A) \,, \tag{C.1}$$
$$Q_j{}^{iM} \equiv -\vartheta_j{}^A Y_A{}^{iM} = -\vartheta_j{}^A (T_{A\,k}{}^i Z^{kM} + T_{A\,N}{}^M Z^{iN}) \,, \tag{C.2}$$
$$Q_{k\,M\,ij} \equiv -\vartheta_k{}^A Y_{A\,M\,ij} = \vartheta_k{}^A (2T_{A\,(i|}{}^l d_{M\,|j)l} + T_{A\,M}{}^N d_{N\,ij}) \,, \tag{C.3}$$

associated to their gauge-invariance and, furthermore, to the constraints

$$Q^{AM} \equiv \vartheta_i{}^A Z^{iM} \,, \tag{C.4}$$
$$Q^{ij} \equiv Z^{iM} Z^j{}_M \,, \tag{C.5}$$
$$Q_{jk}{}^i \equiv X_{(jk)}{}^i - Z^{iM} d_{M\,jk} \,, \tag{C.6}$$
$$Q_{i\,MN} \equiv X_{i\,MN} - 4 Z^j{}_{[M} d_{N]ij} \,, \tag{C.7}$$
$$Q_{ijk\,l} \equiv d_{M\,(ij} d^M{}_{k)l} \,. \tag{C.8}$$

C.2 Field strengths and Bianchi identities

The tensor hierarchies of general 6-dimensional bosonic field theories have 1-forms A^i, 2-forms B_M, 3-forms C_i, 4-forms D_A, three types of 5-forms $E^i{}_A, E_{iM}, E^{M\,ij}$ and eight types of 6-forms (that we will only refer to collectively as $F_\flat$). The field strengths of the 1- to 5-form potentials are given by

$$F^i = dA^i + \frac{1}{2} X_{jk}{}^i A^{jk} + Z^{iM} B_M \,, \tag{C.9}$$

$$H_M \equiv \mathfrak{D} B_M + d_{M\,jk} \left[A^j \wedge dA^k + \frac{1}{3} X_{lm}{}^k A^{jlm} \right] - Z^i{}_M C_i \,, \tag{C.10}$$

$$\begin{aligned} G_i = {} & \mathfrak{D} C_i + 2 d^N{}_{ip} \left[\left(F^p - \frac{1}{2} Z^{pM} B_M \right) \wedge B_N + \frac{1}{3} d_{N\,jk} A^{pj} \wedge dA^k + \frac{1}{12} X_{jk}{}^n d_{N\,ln} A^{pjkl} \right] \\ & + \vartheta_i{}^A D_A \,, \end{aligned} \tag{C.11}$$

$$\begin{aligned} K_A = {} & \mathfrak{D} D_A + T_A{}^{MN} \left(H_M - \frac{1}{2} \mathfrak{D} B_M \right) \wedge B_N \\ & + T_{A\,k}{}^p \left[\left(F^k - Z^{kM} B_M \right) \wedge C_p - \frac{1}{6} d^M{}_{jp} d_{M\,lm} A^{jkl} \wedge dA^m \right. \\ & \left. + \frac{1}{30} X_{lm}{}^q d^M{}_{jq} d_{M\,pn} A^{jklmn} \right] + Y_{A\,i}{}^B E^i{}_B + Y_A{}^{iM} E_{iM} + Y_{A\,M\,ij} E^{M\,ij} \,, \end{aligned} \tag{C.12}$$

$$
\begin{aligned}
L^i{}_B &= \mathfrak{D} E^i{}_B - F^i \wedge D_B + \frac{1}{30} T_{B\,k}{}^n d^N{}_{jm} d_{N\,ln} A^{ijkl} \wedge dA^m \\
&\quad + \frac{1}{80} T_{B\,k}{}^p X_{lm}{}^q d^N{}_{jq} d_{N\,pn} A^{ijklmn} + \frac{\partial Q^\flat}{\partial \vartheta_i{}^B} F_\flat\,, \qquad \text{(C.13)}\\
L_{iM} &= \mathfrak{D} E_{iM} + (H_M - \mathfrak{D} B_M) \wedge C_i - B_M \wedge (G_i - \vartheta_i{}^B D_B) - \frac{1}{2} Z^j{}_M C_{ij} \\
&\quad + d^N{}_{ij} F^j \wedge B_{MN} + \frac{1}{3} d^N{}_{ij} Z^{jP} B_{MNP} + \frac{\partial Q^\flat}{\partial Z^{iM}} F_\flat\,, \qquad \text{(C.14)}\\
L_M{}^{ij} &= \mathfrak{D} E_M{}^{ij} - F^{ij} \wedge B_M + Z^{iN} F^j \wedge B_{MN} - \frac{1}{3} Z^{iN} Z^{jP} B_{MNP} \\
&\quad - \frac{1}{2} d_{M\,kl} A^{ik} \wedge dA^{jl} - \frac{2}{15} X_{kl}{}^n d_{M\,nm} A^{iklm} \wedge dA^j - \frac{1}{5} X_{kl}{}^j d_{M\,nm} A^{ikln} \wedge dA^m \\
&\quad - \frac{1}{18} X_{kl}{}^j X_{np}{}^q d_{M\,mq} A^{iklmnp} + \frac{\partial Q^\flat}{\partial d^M{}_{ij}} F_\flat\,. \qquad \text{(C.15)}
\end{aligned}
$$

These field strengths are related by the following Bianchi identities

$$
\begin{aligned}
\mathfrak{D} F^i &= Z^{iM} H_M\,, \qquad \text{(C.16)}\\
\mathfrak{D} H_M &= d_{M\,ij} F^{ij} - Z^i{}_M G_i\,, \qquad \text{(C.17)}\\
\mathfrak{D} G_i &= 2 d^M{}_{ij} F^j \wedge H_M + \vartheta_i{}^A K_A\,, \qquad \text{(C.18)}\\
\mathfrak{D} K_A &= T_{A\,j}{}^k F^j \wedge G_k - \frac{1}{2} T_A{}^{MN} H_{MN} \\
&\quad + Y_{A\,i}{}^B L^i{}_B + Y_A{}^{iM} L_{iM} + Y_A{}^M{}_{ij} L_M{}^{ij}\,. \qquad \text{(C.19)}
\end{aligned}
$$

C.3 Duality relations

$$
\begin{aligned}
H_\Lambda &= J_\Lambda = b_{\Lambda\Sigma} \star H^\Sigma + c_{\Lambda\Sigma} H^\Sigma\,, \qquad \text{(C.20)}\\
G_i &= a_{ij} \star F^j\,, \qquad \text{(C.21)}\\
K_A &= \star j_A\,, \qquad \text{(C.22)}\\
L_\sharp &= \star \frac{\partial V}{\partial c^\sharp}\,. \qquad \text{(C.23)}
\end{aligned}
$$

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